SPRING 2020: MATH 996 CLASS NOTES STARTING MARCH 23

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1. Nagata rings

In this part of the course we deal with the following question: Given a Noetherian domain R with quotient field K, when is the integral closure of R in a finite extension of K a finite R-module? The rings from algebraic geometry have this property. Our immediate goal is to see to what extent this property holds in a purely algebraic setting.

Until further notice, or unless indicated otherwise, R will denote a Noetherian integral domain with quotient field K. We will consistently use L to denote a finite field extension of K and S to denote the integral closure of R in L. For an integral domain T, we will write T' for the integral closure of T (in its quotient field).

Definition. Maintaining the notation above:

(i) R is said to satisfy N_1 if R' is a finite R-module.
(ii) R is said to satisfy N_2 if S' is a finite S-module for all finite extensions L of K.
(iii) R is said to be a *Nagata ring* if R/P satisfies N_2, for all prime ideals $P \subseteq R$.

Comments. 1. The main goal of this section is to prove that if R is a Nagata ring, then any finitely generated R-algebra T is a Nagata ring.

2. Though we are assuming throughout that R is an integral domain, the definition of Nagata ring clearly applies to any Noetherian ring. The theorem we seek for arbitrary rings reduces trivially to integral domains, so we do not lose any generality by assuming R and T are integral domains.

3. If S is a finite extension of R, then R satisfies N_2 if and only if S satisfies N_2. The forward direction is clear. If S satisfies N_2 and A is the integral closure of R in a finite extension E of K, then A is contained in the integral closure B of S in the field obtained by adjoining E to L. Since B is finite over S, it is finite over R, and thus A is finite over R.

4. If R satisfies N_2 or is a Nagata ring, then R_S is N_2 or a Nagata ring, for any multiplicatively closed set $S \subseteq R$.

5. If S is a finite extension of R, then R is a Nagata ring if and only if R is a Nagata ring. This follows easily from 3 above and the lying over property for integral extensions.

6. If R is a complete local domain, then R is a Nagata ring. Why: Each factor R/P is a complete local domain and complete local domains satisfy N_2 (coming soon).

We will require a number of preliminary results before getting to the main result on Nagata rings. For this, we need a good understanding of integral closure. Our first result is a variation on Serre's criteria for a ring to be integrally closed.

Proposition A. *Let $0 \neq x \in R$. Then xR is an integrally closed ideal if and only if R_P is a DVR, for all $P \in Ass(R/xR)$.*

Proof. Suppose that for each $P \in \text{Ass}(R/xR)$, R_P is a DVR, and fix one such P. If $y \in \overline{xR}$, then $y \in \overline{xR_P} = xR_P$, since R_P is a DVR. Since this holds for all P, $y \in xR$.

Now suppose xR is integrally closed and $P \in \text{Ass}(R/xR)$. We may assume R is local at P. Write $P = (xR : a)$. Then $P \cdot \frac{a}{x}$ is an ideal of R. If $P \cdot \frac{a}{x} \subseteq P$, then by the *determinant trick*, $\frac{a}{x}$ is integral over R. Thus, $a \in \overline{xR} = xR$, a contradiction. Therefore $P \cdot \frac{a}{x} = R$.

Take $p_0 \in P$ such that $p_0 \cdot \frac{a}{x} = 1$. Now take any $p \in P$. Note that $p \cdot \frac{a}{x} \in R$. Thus, $p = (p \cdot \frac{a}{x}) \cdot p_0$, which shows $P = p_0R$. Therefore, R is a DVR. □

Corollary B. *For R as above, R is integrally closed if and only if for every prime $P \subseteq R$ associated to a principal ideal, R_P is a DVR.*

Proof. Since an element $\frac{a}{x} \in K$ is integral over R if and only if $a \in \overline{xR}$, R is integrally closed if and only if each principal ideal xR is integrally closed. Thus, the corollary follows immediately from Theorem A. $\square$

Corollary C. *Suppose there exists $0 \neq x \in R$ such that R_x is integrally closed. Then there exists an ideal $J \subseteq R$ such that for all prime ideals $Q \subseteq R$, R_Q is integrally closed if and only if $J \not\subseteq Q$.*

Proof. If R_P is a DVR for all $P \in \mathrm{Ass}(R/xR)$, then the previous corollary implies that R is integrally closed and we just take $J = R$.

Otherwise, let $P_1, \dots, P_r$ be the prime ideals in $\mathrm{Ass}(R/xR)$ such that R_P is NOT a DVR. Set $J := P_1 \cap \cdots \cap P_r$. Suppose $Q \subseteq R$ is a prime ideal. If $J \subseteq Q$, then $P_i \subseteq Q$, some i. Since R_Q localized at P_iR_Q is just R_{P_i}, xR_Q is not integrally closed, and thus R_Q is not integrally closed.

Suppose R_Q is not integrally closed. Then we must have $x \in Q$ and xR_Q is not integrally closed. By standard localization arguments, $P_i \subseteq Q$, for some i. Thus, $J \subseteq Q$. $\square$

Suppose R is locally analytically unramified and T is a finitely generated R-algebra contained in K. Then T' is locally finite over T. In general, if T is a Noetherian domain and $T'_{\mathfrak{m}}$ is finite over $T_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$, then it need not be the case that T' is finite over T. However, the following important result gives a case when this does hold.

Theorem D. *Let T be a Noetherian domain satisfying the properties:*

(i) T_b is integrally closed for some $0 \neq b \in T$.
(2) $T'_{\mathfrak{m}}$ is finite over $T_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ containing b.

Then T' is finite over T.

Remark. Note that conditions (i) and (ii) above together imply that $T'_{\mathfrak{m}}$ is finite over $T_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$.

Proof. For each maximal ideal $\mathfrak{m}$ containing b, let $T \subseteq T(\mathfrak{m}) \subseteq T'$ be a ring which is a finite T-module satisfying $T(\mathfrak{m})_{\mathfrak{m}} = T'_{\mathfrak{m}}$. Fix $\mathfrak{m}$. Since $T_b = T'_b$, $T(\mathfrak{m})_b = T'_b$ is integrally closed. By the previous corollary, there exists $J(\mathfrak{m}) \subseteq T(\mathfrak{m})$ such that for all primes $Q \subseteq T(\mathfrak{m})$, T_Q is integrally closed if and only if $J(\mathfrak{m}) \not\subseteq Q$.

Set $I(\mathfrak{m}) = J(\mathfrak{m}) \cap T$. Note that since $T(\mathfrak{m})_{\mathfrak{m}}$ is integrally closed, $I(\mathfrak{m}) \not\subseteq \mathfrak{m}$, since $J(\mathfrak{m})_{\mathfrak{m}} = T(\mathfrak{m})_{\mathfrak{m}}$. Let $J \subseteq T$ be the ideal such that T_Q is integrally closed if and only if $J \not\subseteq Q$. Thus $J \not\subseteq \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ not containing b.

If we take the sum of all ideals $I(\mathfrak{m})$ together with J, then we get an ideal not contained in any maximal ideal of T. Thus, this sum equals T. Therefore, a finite set of ideals from this collection sum to R. Call these ideals $J, I(\mathfrak{m}_1), \dots, I(\mathfrak{m}_s)$. Note that it does no harm to include J, even if it is not required.

Set $\widetilde{T} := T[T(\mathfrak{m}_1), \cdots, T(\mathfrak{m}_s)]$, a finite T-module with $T \subseteq \widetilde{T} \subseteq T'$. We claim $\widetilde{T} = T'$. It suffices to show that $\widetilde{T}_Q = T'_Q$ for all maximal ideals $Q \subseteq \widetilde{T}$. Fix a maximal ideal Q.

Set $\mathfrak{m} := Q \cap T$. Then $\mathfrak{m}$ does not contain $J + I(\mathfrak{m}_1) + \cdots + I(\mathfrak{m}_s)$. If $J \not\subseteq \mathfrak{m}$, then $T_{\mathfrak{m}} = T'_{\mathfrak{m}}$. Hence $\widetilde{T}_{\mathfrak{m}} = T'_{\mathfrak{m}}$, and therefore $\widetilde{T}_Q = T'_Q$.

If $I(\mathfrak{m}_i) \not\subseteq \mathfrak{m}$, set $Q_0 := Q \cap T(\mathfrak{m}_i)$. Then $J(\mathfrak{m}_i) \not\subseteq Q_0$. Thus, $T(\mathfrak{m}_i)_{Q_0} = T'_{Q_0}$. Therefore $\widetilde{T}_{Q_0} = T'_{Q_0}$. Since $\widetilde{T}_Q$ and T'_Q are further localizations of $\widetilde{T}_{Q_0} = T'_{Q_0}$, it follows that $\widetilde{T}_Q = T'_Q$, as required. $\square$

Corollary E. *Suppose R is integrally closed and locally analytically unramified. Let $R \subseteq T \subseteq K$ be a finitely generated R-algebra. Then T' is a finite T-module.*

Proof. We can write $T = R[\frac{a_1}{b}, \dots, \frac{a_n}{b}]$, for $a_i, b \in R$. Then, $T_b = R_b$ is integrally closed. On the other hand, let $\mathfrak{m} \subseteq T$ be a maximal ideal and set $Q := \mathfrak{m} \cap R$. Then T'_Q is finite over T_Q, since R_Q is analytically unramified. Thus, $T'_{\mathfrak{m}}$ is finite over $T_{\mathfrak{m}}$, so the result follows from Theorem D. $\square$

Our next result shows that there is no difference between the conditions N_1 and N_2 for rings having characteristic zero.

Theorem F. *Suppose that R has characteristic zero and satisfies condition N_1. Then R satisfies N_2. In particular, if R is integrally closed, and has characteristic zero, then R satisfies N_2.*

Proof. Let L be a finite extension of K and S the integral closure of R in L. If we show that S is a finite R'-module, then since R satisfies N_1, S is a finite R-module. Thus, it suffices to prove the second statement.

We now assume R is integrally closed. Since L is a separable extension of K, we may enlarge L to a Galois extension L' of K. If the integral closure of R in L' is finite over R, then S is finite over R. Thus, without loss of generality, we assume L is Galois over K.

Since L is separable over K, we may write $L = K(\alpha)$, for some $\alpha \in L$. In fact, we may take $a \in S$, such that $L = K(a)$, by clearing denominators in an equation of algebraic dependence for α over K. Recall: Since R is integrally closed, the minimal polynomial $f(x)$ for a over K has coefficients in R. Let $a = a_1, a_2, \ldots a_n$ be the roots of $f(x)$.

Thus, n is the degree of $f(x)$ and every element in L can be written (uniquley) in the form:

$$k_0 1 + k_1 a + \cdots + k_{n-1}a^{n-1},$$

for $k_j \in K$. Let $\sigma_1, \ldots, \sigma_n$ denote the elements of the Galois group of L over K. Set $d := \prod_{i<j}(\sigma_i(a) - \sigma_j(a))^2$, the *discriminant* of $f(x)$. The proof is complete if we show $d \cdot S \subseteq R[a]$.

Let $s \in S$ and write

$$s = k_0 1 + k_1 a + \cdots + k_{n-1}a^{n-1}, \qquad (*)$$

with $k_j \in K$. If we show that $d \cdot k_j \in R$, for all j, then $d \cdot s \in R[a]$. Applying each σ_i to (*), we get an $n \times n$ system of equations of the form

$$\sigma_i(s) = k_0 1 + k_1 \sigma_i(a) + \cdots + k_{n-1}\sigma_i(a)^{n-1}. \qquad (**)$$

This yields a matrix equation $A \cdot \begin{bmatrix} k_0 \\ \vdots \\ k_{n-1} \end{bmatrix} = \begin{bmatrix} \sigma_1(s) \\ \vdots \\ \sigma_n(s) \end{bmatrix}$, where $A = (\sigma_i(a)^j)$.

Let $\tilde{A}$ denote the adjugate of A, so that $\tilde{A} \cdot A = \det(A) \cdot I_n$A. We note: (i) A is a Van der Mond matrix, and thus $\det(A)^2 = d$.

(ii) Each $\sigma_i(s)$ and $\sigma_i(a)^j$ is integral over R, and thus belongs to S (since L is Galois over K).

(iii) Multiplying (**) by $\tilde{A}$ shows the entries of $\tilde{A} \cdot \begin{bmatrix} \sigma_1(s) \\ \vdots \\ \sigma_n(s) \end{bmatrix}$ are integral over R.

Thus each $\det(A) \cdot k_i$ is integral over R. Therefore $d \cdot k_i$ is integral over R. On the other hand, for each i, $\sigma_j(dk_i) = dk_i$, for all j. Thus $dk_i \in K$, for all i. Since each dk_i is integral over R, each $dk_i \in R$, as required. ☐

Important Remark. The crucial point in the proof above is the fact that L is separable over K. Thus the proof of Theorem F shows the following: *If R is an integrally closed Noetherian domain with quotient field K, then the integral closure of R in a finite separable extension of K is a finite R-module.*

The example below due to Nagata constructs a one-dimensional local domain S with infinite integral closure and also a one-dimensional DVR R that does not satisfy N_2. Since an excellent local domain must satisfy N_2, R is not excellent.

Example G. We start with a field K of characteristic $p > 0$ such that $[K : K^p] = \infty$. For example, one can take K to be $\mathbb{Z}_p(U_1, U_2, \ldots)$, where $\{U_i\}$ are algebraically independent over $\mathbb{Z}_p$. We set $T := K[[x]]$ and $R := K^p[[x]][K]$, where x is analytically independent over K. We follow the steps below.

Step 1. For $f = \sum_{i=0}^{\infty} \alpha_i x^i \in T$, $f \in R$ if and only if $[K^p(\{\alpha_i\}) : K^p] < \infty$.

Proof: Suppose $f \in R$. Then we can write $f = g_1 k_1 + \cdots + g_r k_r$, for $g_j \in K^p[[x]]$ and $k_j \in K$. If we write $g_j := \sum_{i=0}^{\infty} \beta_{ij} x^i$, then for all $i \geq 0$, we have $\alpha_i = \beta_{i1}k_1 + \cdots + \beta_{ir}k_r$. It follows that $K^p(\{\alpha_i\}) \subseteq K^p \cdot k_1 + \cdots + K^p \cdot k_r$, and thus $[K^p(\{\alpha_i\}) : K^p] < \infty$.

Conversely, suppose $[K^p(\{\alpha_i\}) : K^p] < \infty$. Let $k_1, \ldots, k_r$ be a basis for $K^p(\{\alpha_i\})$ over K^p. Then for each $i \geq 0$, we have an equation $\alpha_i = \beta_{i1}k_1 + \cdots + \beta_{ir}k_r$, with each $\beta_{ij} \in K^p$. It follows that if we set $g_j := \sum_{i=0}^{\infty} \beta_{ij}x^i$, then $f = g_1k_1 + \cdots + g_rk_r$, and hence $f \in R$.

Step 2. R is a discrete valuation ring.

Proof: It suffices to show that xR is the set of non-units of R (and hence xR is the unique maximal ideal of R) and $\bigcap_{i=1}^{\infty} x^nR = 0$. The second statement follows since $\bigcap_{i=11}^{\infty} x^nT = 0$. For the first statement, suppose $f = \sum_{i=0}^{\infty} \alpha_i x^i \in R$ is a non-unit. We claim $\alpha_0 = 0$. Suppose not. Then f is a unit in T, and hence there exists $g = \sum_{i=0}^{\infty} \beta_i x^i$ such that $fg = 1$. If we solve the resulting system of equations

$$\alpha_0\beta_0 = 1$$
$$\alpha_1\beta_0 + \alpha_0\beta_1 = 0$$
$$\vdots$$

for the β_i, we see that $K^p(\{\beta_i\}) \subseteq K^p(\{\alpha_i\})$, and thus $[K^p(\{\beta_i\}) : K^p] < \infty$, since $f \in R$. Thus, $g \in R$, which is a contradiction, since f is a non-unit in R. Thus, $\alpha_0 = 0$. Therefore, we can write $f = x\tilde{f}$. Since the coefficients of $\tilde{f}$ are the same as the coefficients of f, only shifted by one degree, by Step 1, $\tilde{f} \in R$. Thus, $f \in xR$. Therefore, the non-units of R are contained in xR. Since every element of xR is clearly a non-unit in R, it follows that xR is the set of non-units in R. Thus, R is a discrete valuation ring.

Step 3. T is the x-adic completion of R.

Proof: Every f in T in the limit (in the x-adic topology) of a sequence of polynomials $\{f_n\} \subseteq K[x]$. Each $f_n \in R$. Thus, R is dense in T. Since T is complete in the x-adic topology, T is the completion of R.

Step 4. Take $\beta_0, \beta_1, \ldots \in K$ such that $[K^p(\{\beta_i\}) : K^p] < \infty$ and set $a := \sum_{i=0}^{\infty} \beta_i x^i$, so $a \in T \backslash R$. Then $a^p \in R$ and for an indeterminate Y over R , $Y^p - a^p$ is irreducible over R.

Proof: In T, we have $a = \lim_{n\to\infty} a_n$, where $a_n := \beta_0 + \cdots + \beta_n x^n$. Thus, $\lim_{n\to\infty} a_n^p = a^p$. Since $a_n^p = \beta_0^p + \cdots + \beta_n x^{np}$, it follows that $a^p = \sum_{i=0}^{\infty} \beta_i^p x^{ip} \in R$.

Now, a^p is not a pth power in R, otherwise $a^p = r^p$, for some $r \in R$, and thus $(a - r)^p = 0$, so $a - r = 0$, which gives $a \in R$, a contradiction. Since R is integrally closed a^p is not a pth power in the quotient field of R, so $Y^p - a^p$ is irreducible over R.

Step 5. Set $S := R[a]$. Then $S \cong R[Y]/(Y^p - a^p)$ and S is a one dimensional local domain whose integral closure S' is not a finite S-module.

Proof: From the previous step we know that $Y^p - a^p$ generates a height one prime in the UFD $R[Y]$. Since $Y^p - a^p$ belongs to the kernel of the natural map from $R[Y]$ to $R[a]$, it must generate the kernel. This gives the first statement.

For the second statement, S is integral over R, so it is one-dimensional. Moreover, $h^p \in R$, for all $h \in S$, so S must local. To see this, suppose $Q_1, Q_2 \subseteq S$ are two maximal ideals. Since R is local, $Q_1 \cap R = Q_2 \cap R$. Take $h \in Q_1 \backslash Q_2$. Then $h^p \in Q_1 \cap R = Q_2 \cap R$, so $h^p \in Q_2$. Thus, $h \in Q_2$, a contradiction. Therefore S is a one-dimensional local domain.

We claim $\widehat{S} \cong \widehat{R}[Y]/(Y^p - a^p) = T[Y]/(Y^p - a^p)$. Here the completions of R and S are taken with respect to the x-adic topology, which in each case yields the completion with respect to the respective maximal ideals. Suppose the claim holds. In $T[Y]$, $Y^p - a^p = (Y - a)^p$, which shows that $T[Y]/(Y^p - a^p)$ and hence $\widehat{S}$ is reduced. Thus, S is not analytically unramified. By what we have already shown in class, this implies that S' is not a finite S-module.

For the claim, we tensor the exact sequence

$$0 \to (Y^p - a^p)R[Y] \xrightarrow{i} R[Y] \to S \to 0$$

with $\widehat{R}$ to obtain the exact sequence

$$0 \to (Y^p - a^p)R[Y] \otimes \widehat{R} \xrightarrow{\hat{i}} \widehat{R}[Y] \to \widehat{S} \to 0,$$

where we use the easy-to-check fact that $R[Y] \otimes \widehat{R} = \widehat{R}[Y]$. Since the image of the map $\hat{i}$ is $(Y^p - a^p)\widehat{R}[Y]$, this yields the claim.

Step 6. R does not satisfy N_2. Hence R is a non-excellent discrete valuation ring.

Proof: Since a finite extension of a ring satisfying N_2 must have a finite integral closure, the first statement follows from the previous step. The second statement follows from the fact that an excellent local domain must be a Nagata domain, and hence must satisfy N_2. We will see this later in the semester.

Remark. The example above is a special case of Nagata's example, in that Nagata takes more variables. In other words, he sets $T := K[[x_1, \ldots, x_d]]$ and $R := K^p[[x_1, \ldots, x_d]][K]$, where $x_1, \ldots, x_d$ are analytically independent variables over F. Nagata proves that R is a regular local ring with completion T. When $d = 2$ and $d = 3$, he uses R and T to also construct: (a) An example of a two-dimensional Noetherian domain A and a non-Noetherian ring B such that $A \subseteq B \subseteq A'$ and (b) An example of a three-dimensional Noetherian domain C such that C' is not Noetherian. These examples are relevant because on the one hand, every ring between a one-dimensional Noetherian domain and its quotient field is Noetherian, while on the other hand, the integral closure of any two-dimensional Noetherian domain is Noetherian.

The next important theorem shows that a Nagata local domain is analytically unramified.

Theorem H. Assume that $(R, \mathfrak{m}, k)$ is a local Nagata ring. Then R is analytically unramified.

Proof. We proceed with the following steps.

Step 1. Reduction to the case that R is integrally closed.

Proof. We use the fact that if A is a semi-local ring with maximal ideals $P_1, \ldots, P_c$ and $J = P_1 \cap \cdots \cap P_c$, then the J-adic completion of A is the direct sum of the P_i-adic completions of A. To see this, note that for each n,

$$A/J^n \cong A/P_1^n \oplus \cdots \oplus A/P_c^n.$$

Now use the fact that inverse limits commute with direct sums to conclude $\widehat{A}^J = \widehat{A}^{P_1} \oplus \cdots \oplus \widehat{A}^{P_c}$. Note that in this case, $\widehat{A}^J$ is reduced if and only if each $\widehat{A}^{P_i}$ is reduced.

We apply the foregoing to R'. Since R' is finite over R, R has finitely many maximal ideals and $\widehat{R'} = R' \otimes \widehat{R}$, hence the inclusion $R \otimes \widehat{R} \to R' \otimes \widehat{R}$ shows that $\widehat{R}$ is contained in the completion of R' with respect to $\mathfrak{m}R'$. On the other hand, $\sqrt{\mathfrak{m}R'} =: J$ is the Jacobson radical of R'. Thus, the completions of R' with respect to $\mathfrak{m}R'$ and J are the same. The latter is the direct sum of the completions of R' with respect to $P_1, \ldots, P_c$, where the P_i are the maximal ideals of R'. If each $\widehat{R'}^{P_i}$ is reduced, then $\widehat{R}$ is reduced, which is what we want.

Since each R'_{P_i} is also a Nagata ring and $\widehat{R'}^{P_i} = \widehat{R'_{P_i}}^{P_i}$, it suffices to show each R'_{P_i} is analytically unramified. Thus, we may now assume that R is integrally closed.

Step 2. If $0 \neq x \in R$ and $P \in \operatorname{Ass}(\widehat{R}/x\widehat{R})$ satisfies $R/(P \cap R)$ is analytically unramified, then $(\widehat{R})_P$ is a DVR.

Proof. Set $P_0 := P \cap R$ and localize R at P_0. We note that $P_0 \in \operatorname{Ass}(R/xR)$. If not P_0, contains a non-zero divisor on R/xR, which remains a non-zero divisor on its completion $\widehat{R}/x\widehat{R}$, contrary to our assumption on P. Thus, R_{P_0} is a DVR (by Proposition A) and $P_0 = yR$, for some $y \in R$. Since $\widehat{R}$ is (still) flat over R, y is a non-zerodivisor in $\widehat{R}$. It follows that $P \in \operatorname{Ass}(\widehat{R}/y\widehat{R}) = \operatorname{Ass}(\widehat{R}/P_0\widehat{R})$.

On the other hand, $\widehat{R}/P_0\widehat{R}$ is reduced (by assumption). Thus

$$(\widehat{R}/P)_P = (\widehat{R}/P_0\widehat{R})_P = (\widehat{R}/y\widehat{R})_P.$$

Therefore, P_P is principal, so $\widehat{R}_P$ is a DVR.

Step 3. We prove the theorem by induction on the dimension of R.

Proof. Suppose R has dimension one and take $0 \neq x \in R$. Since R' is finite over R, there exists $k \geq 1$ such that $x^n R' \cap R \subseteq x^{n-k}R$, for all $n \geq k$. Thus $\overline{x^n R} \subseteq x^{n-k}R$, for all $n \geq k$. Since xR is $\mathfrak{m}$-primary, R is analytically unramified by Rees's theorem.

If R has dimension greater than one, then by induction, R/Q is analytically unramified for all non-zero prime ideals $Q \subseteq R$. Fix $0 \neq x \in R$, and take $P \in \operatorname{Ass}(\widehat{R}/x^n\widehat{R})$. By Steps 1 and 2, $\widehat{R}_P$ is a DVR. By

Proposition A, $x^n\widehat{R}$ is integrally closed, for all $n \geq 1$. Since the nilradical of $\widehat{R}$ is contained in the integral closure of every ideal, the nilradical of $\widehat{R}$ is contained in $x^n\widehat{R}$ for all n. Thus the nilradical of $\widehat{R}$ is zero, which completes the proof. □

We need one more component, of independent interest, before we can prove the main result of this section. For this result, we will use the following fact about polynomial rings, whose proof we leave as an exercise. Let $A \subseteq B$ be commutative rings and $f(x) \in B[x]$. Then $f(x)$ is integral over $A[x]$ if and only if each coefficient of $f(x)$ is integral over A. It follows that if A is an integrally closed integral domain, the $A[x]$ is also integrally closed.

Theorem I. *Suppose R satisfies N_2. Then the polynomial ring $R[x]$ also satisfies N_2.*

Proof. Let K denote the quotient field of R and suppose L is a finite extension of $K(x)$, the quotient field of $R[x]$. Let S denote the integral closure of $R[x]$ in L. Clearly, $R'[x] \subseteq S$. If S is a finite $R'[x]$-module, then since $R'[x]$ is a finite $R[x]$-module (R' is finite over R), S will be a finite $R[x]$-module. Thus, we may replace R by R' and assume that R is integrally closed. Then $R[x]$ is also integrally closed.

If R has characteristic zero, the proof is complete, by Theorem F. Suppose R has characteristic $p > 0$, i.e., $\mathbb{Z}_p \subseteq R$. We claim there exists a finite extension K' of K, an exponent $q = p^e$, for some e, and γ in the algebraic closure of L such that $L \subseteq K(x^{\frac{1}{q}}, \gamma)$ and γ is separable over $K'(x^{\frac{1}{q}})$.

Suppose the claim holds. Let R_0 be the integral closure of R in K'. Then $R_0[x]$ is the integral closure of $R[x]$ in $K'(x)$. Since R_0 is finite over R, $R_0[x]$ is finite over $R[x]$. If the integral closure of $R_0[x]$ in $K'(x^{\frac{1}{q}}, \gamma)$ is finite over $R_0[x]$, it is finite over $R[x]$. Thus, we may replace K' by K and R_0 by R and assume $K = K'$.

So, let S denote the integral closure of $R[x]$ in $K(x^{\frac{1}{q}}, \gamma)$. Let T denote the integral closure of $R[x]$ in $K(x^{\frac{1}{q}})$. Since $R[x^{\frac{1}{q}}]$ is contained in $K(x^{\frac{1}{q}})$, is integral over $R[x]$, and is integrally closed (it's a polynomial ring over R), we have $T = R[x^{\frac{1}{q}}]$, which is a finite $R[x]$-module. On the other hand, by the Remark following Theorem F, the integral closure of T in $K(x^{\frac{1}{q}}, \gamma)$, which is S, is a finite T-module. Thus, S is a finite $R[x]$-module, as required.

It remains to prove the claim. For this, we first make an observation. Let E be a field of characteristic $p > 0$ and suppose β is separable over E, with minimal polynomial $f(y)$. If $E_0 \supseteq E$ is a field containing the qth roots of the coefficients of $f(y)$, then $\beta^{\frac{1}{q}}$ is separable over E_0. To see this, suppose $f(y) = y^n + e_1y^{n-1} + \cdots + e_n$, with each $e_j \in E$. If δ is a root of $f(y)$, then $\delta^{\frac{1}{q}}$ is a root of $f_q(y) = y^n + e_1^{\frac{1}{q}}y^{n-1} + \cdots + e_n^{\frac{1}{q}}$. Since $f(y)$ has distinct roots, $f_q(y)$ has distinct roots, and hence $\beta^{\frac{1}{q}}$ is separable over E_0.

To prove the claim, we can write $K(x) \subseteq F \subseteq L$, where F is separable over $K(x)$ and L is purely inseparable over F. There exists $\beta \in F$ such that $F = K(x, \beta)$. Moreover, there exist $\alpha_1, \ldots, \alpha_s \in L$ and $q = p^e$ such that $\alpha_i^q \in K(x, \beta)$, for all i, and $L = K(x, \beta, \alpha_1, \ldots, \alpha_s)$. Suppose

$$f(y) = y^n + \frac{c_n(x)}{d_n(x)}y^{n-1} + \cdots + \frac{c_n(x)}{d_n(x)}$$

is the minimal polynomial of β over $K(x)$. For each $1 \leq i \leq s$, we have an equation

$$\alpha_i^q = \frac{a_{0,i}(x)}{b_{0,i}(x)} \cdot 1 + \cdots + \frac{a_{n-1,i}(x)}{b_{n-1,i}(x)} \cdot \beta^{n-1},$$

where the fractions in this equation belong to $K(x)$. Let K' be the field obtained by adjoining the qth roots of the coefficients of all $a_{i,j}(x), b_{i,j}(x), c_i(x), d_i(x)$ to K. Set $\gamma = \beta^{\frac{1}{q}}$. Then $L \subseteq K'(x^{\frac{1}{q}}, \gamma)$ and K' is a finite extension of K. By the observation above, γ is separable over $K'(x^{\frac{1}{q}})$. The proof of Theorem I is now complete. □

We are now ready to prove the main result of this section.

Theorem J. *Suppose R is a Nagata ring and T is a finitely generated R-algebra. Then T is a Nagata ring.*

Proof. By induction on the number of ring generators of T over R, we may assume that $T = R[x]$, for some $x \in T$. Let $Q \subseteq T$ be a prime ideal. We must show that T/Q satisfies N_2. Set $q := Q \cap R$. Then $T/Q = R/q[\overline{x}]$, where $\overline{x}$ denotes the image of x in T/Q. Since R/q is a Nagata ring, upon changing notation we are reduced to proving the following statement. If R is a Nagata ring and T is an integral domain generated as a ring over R by a single element x, then T satisfies N_2.

If x is algebraically independent over R, then we are done by the previous theorem.

Suppose x is algebraic over R. If x is integral over R, then T is a Nagata ring, by Comment 5 above. Otherwise, there exists $a \in R$ such that ax is integral over R. Thus the ring $A := R[ax]$ is a Nagata ring, by Comment 5 above. Moreover, A and T have the same quotient field and $T = A[x]$. Thus, we may replace A by R and begin again assuming $T = R[x]$ with, $x \in K$, the quotient filed of R.

Let L be a finite extension of K (the quotient field of R and T), write $\tilde{R}$ for the integral closure of R in L and S for the integral closure of T in L. Then $\tilde{R}$ is an integrally closed Nagata ring, $\tilde{R}$ and $\tilde{R}[x]$ have the same quotient field and $\tilde{R}[x]$ is finite over T, since $\tilde{R}$ is finite over R. Moreover, $S = \tilde{R}[x]'$. If we show $\tilde{R}[x]'$ is finite over $\tilde{R}[x]$, then S will be finite over T, which is what we want. But now, $\tilde{R}$ is integrally closed, and by Theorem H, $\tilde{R}$ is locally analytically unramified. Thus $\tilde{R}[x]'$ is finite over $\tilde{R}[x]$ by Corollary E. ☐

We now easily recover the geometric case.

Corollary K. Let k be a field and R a finitely generated k-algebra. Then R is a Nagata ring. In particular, the integral closure of R in a finite extension of its quotient field is a finite R-module.

Proof. k is a Nagata ring!

2. Krull Domains and the Mori-Nagata Theorem

The purpose of this part of the course is to address the degree to which the integral closure of a Noetherian domain fails to be Noetherian. In the previous section, we saw that Nagata's example shows that the integral closure of a one-dimensional Noetherian domain R need not be a finite R-module. It is, however, a Noetherian ring. This will follow from the results below. As mentioned above, the integral closure of a two-dimensional Noetherian domain is again Noetherian, but this fails for Noetherian domains of dimension greater than two. This failure is mitigated by the fact that the integral closure is Noetherian-like in codimension one. This is made precise by saying that the integral closure of a Noetherian domain is a Krull domain, a fact known as the Mori-Nagata theorem. Therefore, the purpose of this part of the course is to prove the Mori-Nagata theorem.

Definition. Let S be an integral domain with quotient field L. We say that S is a *Krull domain* if the following conditions hold.

(i) Each nonzero element of S is contained in only finitely many height one primes.
(ii) S_Q is a DVR, for all height one primes $Q \subseteq S$.
(iii) $S = \bigcap_{\text{height}(Q)=1} S_Q$.

There are a number of ways that a Krull domain behaves like an integrally closed Noetherian domain in codimension one. We illustrate a few of these ways in the proposition below.

Proposition A2. *The following properties hold.*

(a) A Krull domain is integrally closed.
(b) An integrally closed Noetherian domain is a Krull domain.
(c) A Krull domain satisfies the ascending chain condition on principal ideals.
(d) If S is a Krull domain, $0 \neq a \in S$, and $Q_1, \ldots, Q_r$ are the height one prime ideals containing aS, then there exist $e_1, \ldots, e_r \geq 1$ such that $aS = Q_1^{(e_1)} \cap \cdots \cap Q_r^{(e_r)}$ is an irredundant primary decomposition of aS.
(e) If S is a Krull domain and $Q \subseteq S$ is a height one prime, then for any non-zero $a \in Q$, there exists $b \in S$ with $Q = (aS : b)$.

Proof. It is easy to see that an intersection of integrally closed integral domains is integrally closed. Thus (a) follows from (ii) and (iii) in the definition of Krull domain. For (b), Let R be an integrally closed Noetherian domain. Condition (i) holds in R since R is Noetherain, and the height one primes containing a principal ideal aR must be minimal over aR. Condition (ii) holds by Corollary B, since a prime minimal over an ideal is an associated prime of the ideal. Condition (iii) follows from the fact that $R = \bigcap_{P \in \mathcal{P}(R)} R_P$, where $\mathcal{P}(R)$

is the set of prime ideals associated to a non-zero principal ideal, and in case R is integrally closed, $\mathcal{P}(R)$ is just the set of height one prime ideals.

For part (c), let $a_1S \subseteq a_2S \subseteq \cdots$ be an ascending chain of principal ideals. Let Q be a height one prime not containing a_1S. Then $a_1S_Q = S_Q$, and thus $a_nS_Q = S_Q$ for all n. Hence $a_1S_Q = a_nS_Q$ for all n. Now let X be the finite set of height one primes containing a_1S. Take $Q \in X$. Then since S_Q is a DVR, there exists r, depending on Q, such that $a_rS_Q = a_nS_Q$, for all $n \geq r$. Since there are only finitely many primes in X, we can take n_0 the maximum of the r's we just found. It follows that $a_{n_0}S_Q = a_nS_Q$, for all $n \geq n_0$ and all height one primes $Q \subseteq S$. This means $\frac{a_n}{a_{n_0}} \in S_Q$ for all height one primes Q, and so by property (iii) in the definition of Krull domain $\frac{a_n}{a_{n_0}} \in S$, for all $n \geq n_0$. Thus $a_n \in a_{n_0}S$ for all $n \geq n_0$ and therefore the given ascending chain stabilizes at n_0.

For part (d), first recall that if Q is a prime ideal in a commutative ring A, then we define the n^{th} symbolic power of Q to be the ideal $Q^nA_Q \cap A$. Since Q^nA_Q is Q_Q-primary, $Q^{(n)}$ is Q-primary. Now fix a non-zero element $a \in S$ and let $Q_1, \ldots, Q_r$ be the height one prime ideals containing a. Let π_i be the uniformizing parameter for the DVR S_{Q_I} i.e., $\pi_iS_{Q_i} = Q_iS_{Q_i}$, for all i. Then, there exist $e_1, \ldots, e_r$ such that $aS_{Q_i} = \pi_i^{e_i}S_{Q_i} = Q_i^{e_i}S_{Q_i}$, for all i. Thus, $aS \subseteq Q_1^{(e_1)} \cap \cdots \cap Q_r^{(e_r)}$. Now let $x \in Q_1^{(e_1)} \cap \cdots \cap Q_r^{(e_r)}$. Then $x \in aS_{Q_i}$ for all i. Let Q be a height one prime not containing a. Then $aS_Q = S_Q$, and hence $x \in aS_Q$. Thus $x \in aS_Q$, for all height one primes Q in S. In other words, $\frac{x}{a} \in \bigcap_{\text{height}(Q)=1} S_Q = S$. Thus, $x \in aS$, which shows $Q_1^{(e_1)} \cap \cdots \cap Q_r^{(e_r)} \subseteq aS$, which is what we want. Finally, the intersection is irredundant, since the nilradicals of the Q_i are distinct. So for instance, if $aS = Q_2^{(e_2)} \cap \cdots \cap Q_r^{(e_r)}$, then $Q_2^{(e_2)} \cap \cdots \cap Q_r^{(e_r)} \subseteq Q_1$. But then some $Q_i^{(e_i)} \subseteq Q_1$ which implies $Q_i \subseteq Q_1$, a contradiction.

For part (e), Let $Q \subseteq S$ be a height one prime and $0 \neq a \in Q$. Take a primary decomposition of aS as in part (d), and assume $Q = Q_1$. By prime avoidance, we can find $b^* \in Q_2^{(e_2)} \cap \cdots \cap Q_r^{(e_r)} \backslash Q$. Take $b_0 \in Q^{(e-1)} \backslash Q^{(e)}$ and set $b := b^*b_0$. If $c \in Q$, then $cb_0 \in Q^{(e)}$, and thus $cb \in aS$. On the other hand, if $cb \in aS \subseteq Q^{(e)}$, then $cb_0 \in Q^{(e)}$, by the choice of b^*. Thus, $cb_0 \in \pi^eS_Q$, where $\pi S_Q = QS_Q$. Since $b_0 \in \pi^{e-1}S_Q$, we have $c \in \pi S_Q \cap S = Q$, which is what we want. Thus, $Q = (aS : b)$. ☐

Remark. Maintain the notation from part (d) in the Proposition above. Then for the given $a \in S$ as in (d), for all $n \geq 1$, $a^nS_{Q_i} = \pi_i^{ne_i}S_{Q_i}$. Thus, arguing as before, it follows that $a^nS = Q_1^{(ne_1)} \cap \cdots \cap Q_r^{(ne_r)}$ is an irredundant primary decomposition of a^nR, for all $n \geq 1$. Here we are using the fact that aS and a^nS are contained in exactly the same set of height one prime ideals.

Examples. (a) Any UFD is easily seen to be a Krull domain. Thus, for example, if k is a field, the polynomial ring in countably many variables over k is a non-Noetherian UFD, and hence a non-Noetherian Krull domain.
(b) If R is a Krull domain, then a polynomial ring in countable many variables over R is a Krull domain. Thus if $R = K[x, y, z, w]/(xy - zw)$, then adjoining countably many variables yields a non-Noetherian Krull domain that is not a UFD.

The following technical proposition due to J. Nishimura has a number of applications, including the lovely theorem which follows it

Proposition B2. *Let S be a Krull domain and $Q \subseteq S$ a height one prime ideal. Then, for all $n \geq 1$, the S-module $Q^{(n)}/Q^{(n+1)}$ embeds into S/Q.*

Proof. By part (d) of Proposition A2, we can write $Q = (aS : b)$, for $a, b \in S$. If we take a primary decomposition

$$aS = Q^{(e)} \cap Q_2^{(e_2)} \cap \cdots \cap Q_r^{(e_r)},$$

the proof of part (d) shows that we can assume $b \in Q^{(e-1)} \cap Q_2^{(e_2)} \cap \cdots \cap Q_r^{(e_r)}$. We claim that if $x \in Q^{(n)}$, then $x \in (a^nS : b^n)$. Thus, $x \cdot \frac{b^n}{a^n} \in S$.

To see the claim, take $s \in S \backslash Q$ such that $sx \in Q^n$. Then $sxb^n \in a^nS$. Since $Q^{(en)}$ is the Q-primary component of a^nS and $s \notin Q$, $xb^n \in Q^{(en)}$. On the other hand, $b^n \in Q_2^{(ne_2)} \cap \cdots \cap Q_r^{(ne_r)}$, so $xb^n \in a^nS$.

We thus have an S-module map $Q^{(n)} \overset{\frac{b^n}{a^n}}{\to} S \to S/Q$. Call this map ϕ. We need to show that if $x \in Q^{(n)}$, then $\phi(x) \in Q$ if and only if $x \in Q^{(n+1)}$. If so, then ϕ induces an injective map from $Q^{(n)}/Q^{(n+1)}$ into S/Q, as required.

Take $x \in Q^{(n)}$ and assume $\pi \in Q$ satisfies $\pi S_Q = QS_Q$. Suppose $\phi(x) \in Q$. Then $xb^n \in Qa^n$. Therefore, $xb^n \in \pi a^n S_Q$. But in S_Q, $b = u\pi^{e-1}$ and $a \in \pi^e S_Q$, where $u \in S_Q$ is a unit. Thus $x\pi^{n(e-1)} \in \pi^{en+1}S_Q$. It follows that $x \in \pi^{n+1}S_Q \cap S = Q^{(n+1)}$. Conversely, suppose $x \in Q^{(n+1)}$, Then there exists $s \in S \backslash Q$ such that $sx \in Q^{n+1}$. Then $sxb^n \in a^n Q$. Therefore, $s(x \cdot \frac{b^n}{a^n}) \in Q$. In other words, $s \cdot \phi(x) \in Q$. But $\phi(x) \in S$ and $s \notin Q$, so $\phi(x) \in Q$, which is what we want. □

Theorem C2. (Nishimura) *Let S be a Krull domain. If S/Q is Noetherian for all height one primes $Q \subseteq S$, then S is Noetherian.*

Proof. Let $I \subseteq S$ be an ideal and take a non-zero $a \in I$. It suffices to show that I/aS is a finitely generated S-module. Take the primary composition $aS = Q_1^{(e_1)} \cap \cdots \cap Q_r^{(e_r)}$ as above, where the Q_i are the height one primes containing a. Then, on the one hand,

$$S/aS \hookrightarrow S/Q_1^{(e_1)} \oplus \cdots \oplus S/Q_r^{(e_r)},$$

so it suffices to show that $S/Q_1^{(e_1)} \oplus \cdots \oplus S/Q_r^{(e_r)}$ is a Noetherian S-module.

On the other hand, given any height one prime $Q \subseteq S$, our assumption on Q and Proposition B2 show that $Q^{(n-1)}/Q^{(n)}$ is a Noetherian S/Q-module, and hence a Noetherian S-module, for all $n \geq 1$ (since Q annihilates $Q^{(n-1)}/Q^{(n)}$). Thus, induction on n and the short exact sequences

$$0 \to Q^{(n-1)}/Q^{(n)} \to S/Q^{(n)} \to S/Q^{(n-1)} \to 0$$

show that each $S/Q^{(n)}$ is a Noetherian S-module. Therefore, $S/Q_1^{(e_1)} \oplus \cdots \oplus S/Q_r^{(e_r)}$ is a Noetherian S-module, which completes the proof. □

We will need several preliminary results before giving the proof of the Mori-Nagata theorem. We start with the theorem of Matijevic, which generalizes the Krul-Akizuki theorem. For this result we need the notion of the global transform of a Noetherian ring.

Definition. Let R be a Noetherian ring with total quotient ring K. The *global transform of* R is the set T of elements $x \in K$ such that $(R : x)$ has the property that $R/(R : x)$ is Artinian, i.e., $R/(R : x)$ is zero-dimensional. Equivalently, T consists of the set of elements $x \in K$ such that $(R : x)$ contains a product of maximal ideals.

Remarks. 1. It is easy to check that T is a subring of K containing R. In fact, if $x, y \in T$, $Jx \subseteq R$, $Iy \subseteq R$, and I, J each contain a product of (possibly different) maximal ideals, then JI contains a product of maximal ideals and $JIxy \subseteq R$. Thus, $xy \in T$. The proof that $x + y \in T$ is similar.

2. If R has dimension one, then K is the global transform of R. Indeed, if $x = \frac{a}{b} \in K$, then $b \in (R : x)$. Since b is a non-zerodivisor, $\dim(R/bR) = 0$. Thus, $\dim(R/(R : x)) = 0$.

3. Suppose $(R, \mathfrak{m}, k)$ is a local ring. Then the global transform is the set of elements $x \in K$ such that there exists $n \geq 1$ with $\mathfrak{m}^n \cdot x \subseteq R$. This is the so-called *ideal transform* of $\mathfrak{m}$.

4. More generally, if $I \subseteq R$ is an ideal, $T(I)$, the *ideal transform of I*, is the set of elements $x \in K$ such that $I^n x \subseteq R$, for some $n \geq 1$. Note that if $\frac{a}{b}$ belongs to the ideal transform of I, then $I^n a \subseteq bR$, for some n, which means I consists of zerodivisors modulo bR. Hence grade$(I) = 1$. In general, assuming $T(I) \neq R$, then $T(I)$ need not be finite or integral over R, and in particular, it need not be a Noetherian ring. Ideal transforms played a central role in Nagata's construction of a counter-example to Hilbert's 14th problem, which asked whether or not rings of invariants of certain (infinite) linear groups acting on a polynomial ring $k[x_1, \ldots, x_n]$ were finitely generated k-algebras.

5. Let $I \subseteq R$ be a grade one ideal. We can write $I = (x_1, \ldots, x_r)$, with each $x)_i$ a non-zerodivisor. Then $T(I) = R_{x_1} \cap \cdots \cap R_{x_r}$. To see this, if $u \in T(I)$, then there exists $n \geq 1$ such that $x_i^n \cdot u \in R$, for all i, so $u \in R_{x_i}$, for all i. Conversely, if $u \in R_{x_1} \cap \cdots \cap R_{x_r}$, then taking n large enough, we have $x_i^n \cdot u \in R$, for all i. Choosing $c >> n$, shows that $I^n u \subseteq R$, so that $u \in T(I)$.

Here is Matijevic's Theorem.

Theorem D2. *Let R be a Noetherian ring with total quotient ring K and write T for the global transform of R. Then for any ring $R \subseteq A \subseteq T$ and non-zerodivisor $x \in R$, A/xA is a finite R-module. In particular, A/xA is a Noetherian ring.*

Proof. The second statement follows immediately from the first statement. For the first statement, it suffices to show that $A \subseteq Rx^{-n} + xA$, for some $n \geq 1$. For then, as R-submodules of K, we have

$$A/xA \subseteq (Rx^{-n} + xA)/xA \cong Rx^{-n}/(Rx^{-n} \cap xA),$$

so that A/xA is a submodule of a cyclic module. Fix $a \in A$. We first show there exists $k \geq 1$ such that $a \in Rx^{-k} + xA$. Let $J := (R : a)$, so that R/J is Artinian. Then the images of the ideals (x^n) in R/J form a descending sequence, which ultimately stabilizes. Thus, there exists $k \geq 1$ such that $(x^k) + J = (x^{k+1}) + J$. Write $x^k = rx^{k+1} + j$, for some $r \in R$, $j \in J$. Multiplying by a gives $ax^k = arx^{k+1} + aj$, and hence $a = arx + ajx^{-k}$. But $aj \in R$, so $a \in Ax + Rx^{-k}$, as required.

Now consider the module $(xA \cap R)/xR$. On the one hand, it is generated by the images in R/xR of finitely many elements of the form a_ix. On the other hand, there is a product J of maximal ideal such that $Ja_i \in R$, for all i. Thus, $(xA \cap R)/xR$ is a finitely generated module annihilated by a zero-dimensional ideal, and must therefore have finite length, i.e., $(xA \cap R)/Rx$ is Artinian. Thus, the descending sequence of submodules $(x^hA \cap R, Rx)/Rx$ stabilizes. Thus, in R, the descending sequence of ideals $I_h = (x^hA \cap R, xR)$ stabilizes, at say, $h = n$. We claim $A \subseteq Rx^{-n} + xA$.

Suppose $a \in A$ does no belong to $x^{-n}R + xA$. From the first paragraph, there exists k with $a \in Rx^{-k} + xA$. Note $k > n$. Choose k minimal with this property and write $a = rx^{-k} + xa'$, where $r \in R$ and $a' \in A$. Then $ax^k = r + x^{k+1}a'$. Then $x^k(a - xa') = r \in I_k = I_{k+1}$. Thus, we can write $x^k(a - xa') = x^{k+1}a'' + r'x$. Dividing by x^k, we have $a - xa' = xa'' + r'x^{-k+1}$. Thus, $a = x(a'' - a') + r'x^{-k+1}$. This means, $a \in Rx^{-k+1} + xA$, contradicting the choice of k. Therefore, $A \subseteq Rx^{-n} + xA$, as required. □

Corollary E2. *Let R be a Noetherian domain with global transform T. Then any ring $R \subseteq A \subseteq T$ is Noetherian.*

Proof. Let $I \subseteq A$ be a non-zero ideal. Take $0 \neq x \in I \cap R$. Then A/xA is Noetherian, so I/xA is finitely generated. Thus, I is finitely generated. □

Corollary F2 (Krull-Akizuki.) *Let R be a one-dimensional Noetherian domain with quotient field K. Let L be a finite field extension of K. Then any ring $R \subseteq A \subseteq L$ is Noetherian. Moreover, for any prime ideal (necesarily a maximal ideal) $Q \subseteq A$ and $P = Q \cap R$, $[A/Q : R/P] < \infty$.*

Proof. For the first statement, since L is finite over K, there exists a finite R-module $R \subseteq R_0 \subseteq A$ such that R_0 has quotient field L. Hence R_0 is a one-dimensional Noetherian domain and A is contained in the global transform of R_0, so A is Noetherian by the previous corollary.

For the second statement, take a non-zero element $x \in P$. Then A/xA is a finite A_0-module. It follows that A/Q is finite over A and hence finite over A/Q_0, for $Q_0 := Q \cap A_0$. Since A_0 is finite over R, A_0/Q_0 is finite over R/P. Since $[A/Q : R/P] = [A/Q : A_0/Q_0] \cdot [A_0 : R/P]$, the proof is complete. □

The following corollary of the Krull-Akizuki theorem is very useful and plays a central role in the theory of the integral closure of ideals in Noetherian rings.

Corollary G2. *Let R be a Noetherian domain with quotient field K. Given a non-zero prime ideal $P \subseteq R$, there exists a DVR $(V, \mathfrak{m}_V)$ with $\mathfrak{m}_V \cap R = P$.*

Proof. Without loss of generality, we may localize at P and assume it is the unique maximal ideal of R. Suppose $P = (a_1, \ldots, a_n)R$ and set $T := R[\frac{a_2}{a_1}, \ldots, \frac{a_n}{a_1}]$. Note $PT = a_1T$. Let $Q \subseteq T$ be a height one prime containing a_1T. Then $Q \cap R = P$. T_Q is a one-dimensional local domain, so by the Krull-Akizuki theorem, T'_Q is Noetherian. Take a maximal ideal $\mathfrak{m} \subseteq T'_Q$ lying over QT_Q. Then $V = (T'_Q)_{\mathfrak{m}}$ is a DVR and its maximal ideal $\mathfrak{m}_V$ has the property that $\mathfrak{m}_V \cap T = Q$. Thus, $\mathfrak{m}_V \cap R = P$, as required.

Remark. Corollary E2 holds if R is just a reduced Noetherian ring with total quotient ring K. Take $R \subseteq A \subseteq T$. Since K is a localization of R, it is not difficult to see that A has finitely many minimal primes, say $Q_1, \ldots, Q_r$ and they are all of the form $QK \cap A$, for Q a minimal prime of R. Let $Q_i \subseteq A$ be a minimal prime. Then $Q_i \cap R$ is a minimal prime. The ring A/Q_i lies between $R/(Q_i \cap R)$ and its global transform, and thus A/Q_i is Noetherian. As such, it is also a Noetherian A-module, since the action of A on A/Q_i is the same as the action of A/Q_i on itself. Since $A \hookrightarrow A/Q_i \oplus \cdots \oplus A/Q_r$, it follows that A is a Noetherian A-module, and hence a Noetherian ring.

We want to present one more application of Matijevic's theorem that applies to ideal transforms - even though it is not related to the Mori-Nagata theorem. This result shows that for local rings $(R, \mathfrak{m})$ with well behaved completions, the transform $T(\mathfrak{m})$ is a finite R-module. In some sense, this is not saying too much, because if R has depth greater than one, $T(\mathfrak{m}) = R$. For this result we need the lemma below and the following standard fact we leave as an exercise. If $(R, \mathfrak{m})$ is a complete local ring, then R is complete in the I-adic topology, for any ideal $I \subseteq R$. We note that the condition on the completion of R in Theorem I2 will always hold for a local ring from algebraic geometry that is reduced and equidimensional, e.g., an integral domain.

Lemma H2. *Let $(R, \mathfrak{m})$ be a local ring. Then $T(\mathfrak{m})$ is a finite T-module if and only if $T(\mathfrak{m}\widehat{R})$ is a finite $\widehat{R}$-module.*

Proof. By Remark 5 after the definition of the global transform, $T(\mathfrak{m}) = R_{x_1} \cap \cdots \cap R_{x_r}$. Therefore,

$$T(\mathfrak{m}) \otimes \widehat{R} = R_{x_1} \cap \cdots \cap R_{x_r} \otimes \widehat{R} = \widehat{R}_{x_1} \cap \cdots \cap \widehat{R}_{x_r} = T(\mathfrak{m}\widehat{R}),$$

since tensoring with a faithfully flat extension distributes over a finite intersection. Thus, $T(\mathfrak{m}\widehat{R}) = T(\mathfrak{m}) \otimes \widehat{R}$. By faithful flatness, $T(\mathfrak{m})$ is finite over R if and only if $T(\mathfrak{m}\widehat{R})$ is finite over $\widehat{R}$. □

Theorem I2. *Let $(R, \mathfrak{m})$ be a local ring with positive depth. Then $T(\mathfrak{m})$ is a finite R-module if and only if there does not exist $z \in Ass(\widehat{R})$ with $dim(\widehat{R}/z) = 1$.*

Proof. By the previous lemma, we may assume R is complete. Suppose T is not finite over R and take $x \in R$ be a non-zerodivisor. By Matijevic's theorem, T/xT is finite over R. Thus, it must be the case that $\bigcap_{n \geq 1} x^n T \neq 0$. Let $a \in R$ be a not-zero element in $\bigcap_{n \geq 1} x^n T$. Then $\frac{a}{x^n} \in T$, for all $n \geq 1$. Thus, for each $n \geq 1$, there exists $s(n) \geq 1$ with $\mathfrak{m}^{s(n)} \cdot \frac{a}{x^n} \subseteq R$. In other words, $\mathfrak{m}^{s(n)} \subseteq (x^n R : a)$, for all n. However, there exists $k \geq 1$ such that $(x^n : a) \subseteq (0 : a) + x^{n-k}R$, for $n \geq k$ (a consequence of Artin-Rees). If $z \in \text{Ass}(R)$ contains $(0 : a)$, and $n = k + 1$, we have $\mathfrak{m}^{s(n)} \subseteq x + zR$. Thus, $\mathfrak{m}$ is minimal over $x + zR$, so $\dim(R/z) = 1$, by Krull's principal ideal theorem.

Conversely, suppose $\dim(R/z) = 1$, for $(0 : a) = z \in \text{Ass}(R)$. Take $x \in R$ a non-zerodivisor. Then for all $n \geq 1$, there exists $s(n)$ such that $m^{s(n)} \subseteq x^n R + (0 : a)$. Therefore, $\mathfrak{m}^{s(n)} a \subseteq x^n$, for all n, and thus $\frac{q}{x^n} \in T(\mathfrak{m})$, for all $n \geq 1$. But then $R \cdot \frac{a}{x} \subsetneq R \cdot \frac{a}{x^2} \subsetneq \cdots$ is a strictly increasing chain of submodules of $T(\mathfrak{m})$. Therefore, $T(\mathfrak{m})$ is not finite over R. □

The following corollary is for those who are familiar with local cohomology.

Corollary J2. *Let $(R, \mathfrak{m})$ be a local ring having depth one. Thus $H^1_{\mathfrak{m}}(R) \neq 0$. Then $H^1_{\mathfrak{m}}(R)$ is finite if and only if if there does not exist $z \in Ass(\widehat{R})$ with $dim(\widehat{R}/z) = 1$.*

Proof. Set K to be the total quotient ring of R. Note that $\mathfrak{m}K = K$, so $H^i_{\mathfrak{m}}(K) = 0$ for all i. The short exact sequence $0 \to R \to K \to K/R \to 0$ gives rise to the long exact sequence in cohomology

$$\cdots \to H^0_{\mathfrak{m}}(K) \to H^0_{\mathfrak{m}}(K/R) \to H^1_{\mathfrak{m}}(R) \to H^1_{\mathfrak{m}}(K) \to \cdots,$$

Since $H^0_{\mathfrak{m}}(K) = H^1_{\mathfrak{m}}(K) = 0$, we have that $H^0_{\mathfrak{m}}(K/R)$ is isomorphic to $H^1_{\mathfrak{m}}(R)$. But $H^0_{\mathfrak{m}}(K/R)$ is just $T(\mathfrak{m})/R$. Thus, $H^1_{\mathfrak{m}}(R)$ is a finite R-module if and only if $T(\mathfrak{m})/R$ is a finite R-module. But this latter module is finite over R if and only if $T(\mathfrak{m})$ is finite over R. Thus, Theorem I2 implies that $H^1_{\mathfrak{m}}(R)$ is a finite R-module if and only if there does not exist $z \in \text{Ass}(\widehat{R})$ with $\dim(\widehat{R}/z) = 1$, which completes the proof. □

We now turn our attention to the Mori-Nagata Theorem. The proof is based upon ideas of Nagata, Rees, Querre, and McAdam. The original proof due to Nagata used the Cohen structure theorem and properties of completions; in particular, the fact that a complete local domain is finite over a complete regular local ring. More modern treatments, like the one below, avoid the use of completions.

The following is our first lemma. A crucial point in Nagata's proof of the Mori-Nagata theorem was that height one primes in the integral closure of a Noetherian domain R contract to grade one primes in R. His original proof was difficult and required passage to the completion. McAdam gave an elementary proof of this fact. In the lemma below, we adjust McAdam's argument so that it applies to primes in the integral closure minimal over colon ideals. This gives us considerably more mileage. Of course, after the fact, such primes are indeed height one primes.

Lemma K2. *Let R be a Noetherian domain with integral closure S. Take $a, b \in R$, assume that $Q \subseteq S$ is minimal prime over $(aS :_S b)$ and set $P := Q \cap R$. Then P is an associated prime of R/aR.*

Proof. We may assume R is local at P. Since Q is minimal over $(a :_S b)S$, for all $q \in Q$, there exists $s \in S \backslash Q$ such that $s \cdot q^h b \in aS$, for some h. If we do this for the finitely many generators of P, it follows that there exists $s \in S \backslash Q$, $t \geq 1$ and a ring $R \subseteq R_0 \subseteq S$, such that $P^t \cdot sb \subseteq aR'$ and R_0 is a finite R-module. Thus, for all $n \geq 1$, $P^{nt} \cdot s^n b^n \subseteq a^n R_0$. Let $0 \neq c \in R$ satisfy $c \cdot R_0 \subseteq R$. Then, $P^{nt} \cdot (cs^n b^n) \subseteq a^n R$, for all n. If $cs^n b^n \in a^n R$, for all n, then $R[\frac{sb}{a}] \subseteq R \cdot \frac{1}{c}$. This implies that $\frac{sb}{a} \in S$, which implies $s \in (aS :_S b) \subseteq Q$, a contradiction. Thus, for some n, $cs^n b^n \notin a^n R$. Therefore, P^{nt} consists of zero divisors modulo $a^n R$. Since P is maximal, it follows that $P \in Ass(R/a^n R)$, and hence $P \in Ass(R/aR)$, which gives what we want. □

Lemma L2. *Let R be a Noetherian domain and set $A := S \cap T$, where T is the global transform of R and S is the integral closure of R. If $P \subseteq A$ is a maximal ideal and P is an associated prime of a principal ideal, then A_P is a DVR.*

Proof. Suppose $P = (aA : b)$ is maximal. If $\mathfrak{m} := R \cap P$, then $\mathfrak{m}$ is maximal (since A is integral over R) and $\mathfrak{m}(b/a)$ is contained in $A \subseteq T$, so $J\mathfrak{m}(b/a) \in T$, for $J \subseteq R$ a product of maximal ideals. Thus, $b/a \in T$. Now, either $P \cdot P^{-1} = P$ or $P \cdot P^{-1} = A$. In the first case, we get $P \cdot b/a \subseteq P$, which would implies b/a is integral over R and thus $b/a \in S$. But then $b/a \in A$, contradiction. Thus, P is invertible, so P_P is principal. i.e., A_P is a DVR. □

Corollary M2. *Let R be a Noetherian domain with integral closure S. Let $a, b \in R$ and suppose that $Q \subseteq S$ is a prime ideal minimal over $(aS :_S b)$. Set $P := Q \cap R$. Then Q has height one and there are only finitely many height one prime ideals $Q := Q_1, \ldots, Q_h$ in S lying over P. Morover, for any $1 \leq i \leq h$, S_{Q_i} is a DVR.*

Proof. We may assume that R is local at P. Let A be as in the Lemma E2. By the first lemma, applied to the ring A, $Q \cap A$ is associated to a principal ideal. Thus, by Lemma E2, $A_{Q \cap A}$ is a DVR. It follows that $A_{Q \cap A} = S_Q$. In particular, Q has height one. Now the same argument applies to any height one prime in S lying over P. Thus, these all contract to distinct primes in A containing PA. Moreover, since each contraction to A is minimal over PA (by lying over), these contractions are finite in number since A is Noetherian. Thus, only finitely many height one primes in S contract to P. □

We now state and prove the Mori-Nagata theorem :

Theorem N2. *Let R be a Noetherian integral domain with quotient field K and let L be a finite algebraic extension of K. Write S for the integral closure of R in L. Then :*

(1) S is a Krull domain.
(2) For every prime ideal $P \subseteq R$, there are only finitely many primes $Q \subseteq S$ lying over P. Moreover, for any such Q, $[k(Q) : k(P)] < \infty$.

Proof. For (1), we first reduce to the case that $K = L$. Indeed, since L is finite over K, we may find a subring R_0 of S with the following properties : R' is finite over R and R_0 has quotient field L. Thus, S is the integral closure of R_0. Changing notation, we may start again, assuming simply that S is the integral cosure of R.

We now check off the properties required in verifying that S is a Krull domain. First, let $Q \subseteq S$ be a height one prime ideal. Take $0 \neq a \in Q \cap R$. Then Q is minimal over $(a :_S 1)$, so by Corollary C, S_Q is a DVR.

Second, let $0 \neq s \in S$. If we show that some multiple of s is contained in only finitely many height one primes in S, then the same holds for s. Thus, we take $a \in R$, any non-zero multiple of s in R. If Q is a height one prime containing a, then by Lemma D2, $P := Q \cap R$ is an associated prime of R/aR. By Corollary F2 only finitely many height one primes in S lie over P. Since R/aR has only finitely many associated primes, there can only be finitely many height one primes containing aS.

Finally, suppose that $x \in \bigcap S_Q$, where the intersection ranges over the height one primes of S. We can write $x := b/a$, for $b, a \in R$. If x is not in S, then $(aS :_S b)$ is a proper ideal. Let Q be a minimal prime over $(aS :_S b)$. By Corollary F2, Q has height one. But $x \in S_Q$, contradiction. Thus, S is the intersection of its localizations at height one primes. So, S is a Krull domain.

For statement (2) in the theorem, let $P \subseteq R$ be a prime ideal. We may assume R is local at P. Let $Q \subseteq S$ lie over P. We first show by induction on the height of P that $[k(Q) : k(P)] < \infty$. When P has height one, R has dimension one, so we can apply Krull-Akizuki. Suppose the height of P is greater than one. If Q has height one, we let A be as before. As in Corollary F2, $A_{Q\cap A} = S_Q$, so $k(Q) = k(Q \cap A)$. But now, if a is any non-zero element in P, A/aA is finite over R, so $A/Q \cap A$ is finite over R/P, which then gives $[k(Q \cap A) : k(P)] < \infty$. Suppose Q has height greater than one. Then we take Q' properly contained in Q and $P' := Q' \cap R$. By induction applied to P', the quotient field of S/Q' is finite over the quotient field of R/P'. Thus, induction applied to P/P' shows that $[k(Q/Q') : k(P/P')] < \infty$. But, $k(Q/Q') = k(Q)$ and $k(P/P') = k(P)$, so, $[k(Q) : k(P)] < \infty$.

Finally, for P as in the preceding paragraph, we show that there are only finitely many primes $Q \subseteq S$ lying over P. First note that by Corollary 2F, there are only finitely many height one primes in S lying over P. This also completes the proof if the height of P is one, since any prime Q lying over a height one prime has height one. Now, let $0 \neq a$ belong to P. On the one hand, since S is a Krull domain, there are only finitely many minimal primes in S, all of height one, containing aS. On the other hand, any prime Q of height greater than one which contracts to P must contain one of these minimal primes. Let Q' be a height one prime containing aS and set $P' := Q' \cap R$. By the preceding paragraph, the quotient field of S/Q' is finite over the quotient field of R/P'. Thus, by induction on the height of P, in the integral closure S_0 of R/P' in $k(Q')$, there are only finitely many primes lying over P/P'. It follows that S/Q' contains only finitely many primes lying over P/P' (since any such prime lifts to a prime in S_0 lying over P/P'). Thus, there are only finitely many primes in S containing Q' lying over P. Since this holds for each of the finitely many minimal primes of aS, we conclude that there are also only finitely many primes of height greater than one in S lying over P. This completes the proof of the Mori-Nagata theorem. □

As applications of the Mori-Nagata theorem, we will prove that the integral closure of a two-dimensional Noetherian domain is Noetherian and that a complete local domain satisfies N_2.

Theorem O2. *Let R be a two-dimensonal Noetherian domain with quotient field K. Let S be the integral closure of R in a finite extension of its quotient field. Then S is Noetherian.*

Proof. By the Mori-Nagata theorem, S is a Krull domain, so by Nishimura' s theorem (Theorem C2), it suffices to prove that S/Q is Noetherian for all height one primes $Q \subseteq S$. Take such a prime and set $P = Q \cap R$. Note that R/P is a one-dimensional Noetherian domain with quotient field $k(P)$. By the second part of the Mori-Nagata theorem, $[k(Q) : k(P)]$ is finite. Since $R/P \subseteq S/Q \subseteq k(Q)$, S/Q is Noetherian, by the Krull-Akizuki theorem, and the proof is complete. □

The standard proof of the next theorem appeals to the Cohen Structure Theorem. The proof below avoids the use of Cohen' s Structure Theorem, and is modeled on the proof of Nishimura' s Theorem C2.

Theorem P2. *Let $(R, \mathfrak{m})$ be a complete local domain with quotient field K. Let L be a finite extension of K and S be the integral closure of R in L. The S is a finitely generated R-module. In other words, R satisfies the Nagata condition N_2.*

Proof. We will use the fact that if M is an R module and M satisfies the two conditions: (i) M/PM is finite over R and (ii) $\bigcap_{n\geq 1} \mathfrak{m}^n M = 0$, then M is finite over R.

Now, we induct on the dimension of R. Suppose R has dimension one and let a be any non-zero element in $\mathfrak{m}$. Then by the Krull-Akizuki theorem (or Matijevic's theorem), S/aS is finite over R. Thus, $S/\mathfrak{m}S$ is finite over R. On the other hand, S is a Noetherian domain, so $\bigcap_{n\geq 1} \mathfrak{m}^n S = 0$. Thus, S is a finite R-module.

Suppose that R has dimension greater than one. Let $Q \subseteq S$ be a height one prime. Then by the Mori-Nagata theorem, $[k(Q) : k(Q \cap R)] < \infty$, so by induction applied to the complete local domain $R/Q \cap R$, S/Q is a finite $R/Q \cap R$-module, and thus is finite over R. By Nishimura's Proposion B2, $Q^{(n)}/Q^{(n+1)}$ is finite over R for all n. Thus, induction applied to the exact sequences $0 \to Q^{(n)}/Q^{(n+1)} \to S/Q^{(n+1)} \to S/Q^{(n)} \to 0$ shows that $S/Q^{(n)}$ is finite over R, for all $n \geq 1$.

Now, let a be any non-zero element of $\mathfrak{m}$. Since S is a Krull domain, aS has a primary decomposition $aS = Q_1^{(n_1)} \cap \cdots \cap Q_h^{(n_h)}$, where the Q_i are height one primes in S and each $n_1 \geq 1$. Since S/aS embeds into $S/Q_1^{(n_1)} \oplus \cdots \oplus S/Q_h^{(n_h)}$, it follows that S/aS is finite over R. Thus, $S/\mathfrak{m}S$ is finite over R.

Finally, to see that $\bigcap_{n\geq 1} \mathfrak{m}^n S = 0$, let $R_0 \subseteq S$ be finite over R and birational to S and take P a prime (in fact, its the only one) lying over $\mathfrak{m}$. Then there exists a DVR $(V, \mathfrak{m}_V)$ in L with $R_0 \subseteq V$ and $\mathfrak{m}_V \cap R' = P'$. Since V is integrally closed, $S \subseteq V$. Thus, $\mathfrak{m}^n S \subseteq \mathfrak{m}_V^n$, for all n. Since $\bigcap_{n\geq 1} \mathfrak{m}_v^n = 0$, it follows that $\bigcap_{n\geq 1} \mathfrak{m}^n S = 0$, which is what we want. Therefore, S is finite over R. □

3. Quasi-unmixedness and Ratliff's Theorem

The purpose of this section is to study quasi-unmixed local rings with the goal of proving a fundamental theorem due to Ratliiff, which gives equivalent conditions for a local ring $(R, \mathfrak{m})$ to be quasi-unmixed. Recall that R is said to be *quasi-unmixed* or *formally equi-dimensional* if $\dim(\widehat{R}/q) = \dim(\widehat{R})$, for all minimal primes $q \subseteq \widehat{R}$. Here is Ratliff's Theorem, stated for integral domains.

Theorem A3. *Let $(R, \mathfrak{m})$ be a local integral domain. The following statements are equivalent.*

(i) R is quasi-unmixed.
(ii) R is universally catenary.
(iii) R satisfies the dimension formula.

To address the other conditions in Ratliff's theorem, we need a few definitions.

Definition. Let S be a Noetherian ring.
(i) S is *catenary* if for all pairs of primes $P \subseteq Q \subseteq S$, all saturated chains of prime ideals between P and Q have the same length.
(ii) S is *universally catenary* if every finitely generated S-algebra is catenary.
(iii) If S is an integral domain, then S satisfies the *dimension formula* if for every finitely generated S-algebra T and prime ideal $Q \subseteq T$, we have:

$$\text{height}(Q) + \text{tr.deg}_{k(Q\cap S)} k(Q) = \text{height}(Q \cap S) + \text{tr.deg}_S T.$$

Several remarks are in order.

Remarks. (i) The conditions in Ratliff's theorem are not equivalent if R is an arbitrary local ring - for trivial reasons. For example, the ring $k[[x, y, z]]/(x) \cap (y, z)$ is a complete local ring and therefore is universally catenary, something we will see later in this section. On the other hand it is not equi-dimensional and since it is complete, it is not quasi-unmixed. If we assume that R is equi-dimensional, then conditions (i) and (ii) in Ratliff's theorem are equivalent. But the proof of this equivalence easily reduces to the domain case.

(ii) It turns out that the rings from algebraic geometry are all universally catenary. In the late 1940s and early 1950s, it was not known whether or not Noetherian rings in general were catenary or universally catenary. In the mid 1950s, Nagata gave an example of a Noetherian ring that was catenary, but not universally catenary.

(iii) If $S \subseteq T$ an extension of Noetherian domains and T is a finitely generated algebra over S, then the following *dimension inequality* always holds:

$$\text{height}(Q) + \text{tr.deg}_{k(Q\cap S)} k(Q) \leq \text{height}(Q \cap S) + \text{tr.deg}_S T.$$

(iv) To invoke the dimension formula, one needs an extension of integral domains. One could make a definition in the case that S is not a domain, by requiring that S/q satisfies the dimension formula for all minimal primes $q \subseteq S$. Again, in order to have conditions (i) and (iii) in Ratliff's theorem equivalent, one would have to require that S be equi-dimensional, and this case too reduces easily to the case that S is an integral domain.

(v) It is not difficult to see that if S is a Noetherian domain, then the dimension formula holds between S and $S[x]$, the polynomial ring over S. To see this, note that $\text{tr.deg}_S S[x] = 1$. Take a prime $Q \subseteq S[x]$ and set $P = Q \cap S$. There are two cases to consider. If $Q = PS[x]$, then $\text{height}(P) = \text{height}(Q)$ and $S[x]/Q = S/P[x]$, and thus $\text{tr.deg}_{S/P} S[x]/Q = 1$. So the dimension formula holds between S and $S[x]$. If

$Q \neq PS[x]$, then height$(Q) = 1+$height(P) and $S[x]/Q$ is algebraic over S/P, so again, the required equality holds. [1]

We now work towards a characterization of quasi-unmixed local rings obtained by studying *asymptotic sequences*, an integral closure analogue of regular sequences. However, our first goal is to show that if $I \subseteq R$ is an ideal, then $\bigcup_{n\geq 1} \mathrm{Ass}(R/\overline{I^n})$ is finite. Throughout the remainder of this section, R denotes a Noetherian ring.

Lemma B3. *Let S be a Noetherian ring and $J \subseteq S$ be an ideal. Then for $a \in S$, $a \in \overline{J}$ if and only if for all minimal primes $q \subseteq S$, the image of a in S/q belongs to $\overline{(J+q)/q}$.*

Proof. The forward direction is clear. Suppose the image of a in S/q belongs to $\overline{(J+q)/q}$, for all minimal primes $q \subseteq S$. Then for each q there is an n (depending on q) and equation of the form

$$a^n + j_1a^{n-1} + \cdots + j_n \equiv 0 \bmod q,$$

where each $j_i \in J^i$. Taking the product of these equations yields an equation of the form

$$a^m + j_1a^{m-1} + \cdots + j_m \equiv 0 \bmod N,$$

where each $j_i \in J^i$, and N denotes the nilradical of S. Raising this last congruence to an appropriate power shows $a \in \overline{J}$. □

Corollary C3. *Let S be a Noetherian ring and $J \subseteq S$ be an ideal. If $P \in Ass(S/\overline{J})$, then there is a minimal prime $q \subseteq P$ with $P/q \in Ass\ S/\overline{J+q}$.*

Proof. Without loss of generality, we assume S is local at P. Write $P = (\overline{J} : a)$, for some $a \notin \overline{J}$. Then, by Lemma B2, $a \notin \overline{(J+q)/q}$, for some minimal primes $q \subseteq S/q$. Thus, P/q consists of zerodivisors mod $\overline{(J+q)/q}$, which gives what we want. □

Remark. The previous corollary works, even if J has height zero. If J is nilpotent, then $\overline{A^*}(J)$ is just the set of minimal primes, so that $q = P$ and J both become zero modulo P, so the conclusion is trivially true. If J is not nilpotent, then q in the corollary is one of the minimal primes not containing J, and in R/q, the image of J has height greater than zero.

The next crucial proposition is a nice application of the Mori-Nagata theorem and properties of Krull domains.

Proposition D3. *Let S be a Noetherian domain and $0 \neq a \in S$. Then $P \in Ass\ (S/\overline{a^n})S$ for some $n \geq 1$ if and only if there exists a height one prime $Q \subseteq S'$ containing a such that $Q \cap S = P$. In particular $\bigcup_{n\geq 1} Ass\ (S/\overline{a^nS})$ is a finite set.*

Proof. The second statement follows immediately from the first. To prove the first statement, we may assume R is local at P.

Suppose $P \in \mathrm{Ass}\ (S/\overline{a^nS})$ for some $n \geq 1$ and write $P = (\overline{a^nS} : b)$, with $b \notin \overline{a^nS}$. Let $Q_1, \ldots, Q_r$ be the height one primes in the Krull domain S' containing a, and write $a^nS' = C_1 \cap \cdots \cap C_r$, where each C_i is Q_i-primary. Since $a^nS' \cap S = \overline{a^nS}$, $b \notin C_i$, some i. But $Pb \subseteq a^nS' \subseteq C_i$, so we must have $P \subseteq Q_i$, since C_i is Q_i-primary. Therefore $Q_i \cap S = P$.

The proof of the converse requires just minor tweaking of the proof of Lemma K2. Take a height one prime $Q \subseteq S'$ containing a. Since Q is minimal over aS', for all $q \in Q$, there exists $s \in S'\backslash Q$ such that $s \cdot q^h \in aS'$, for some h. If we do this for the finitely many generators of P, it follows that there exists $s \in S'\backslash Q$, $t \geq 1$ and a ring $S \subseteq S_0 \subseteq S'$, such that $P^t \cdot s \subseteq aS_0$ and S_0 is a finite S-module. Thus, for all $n \geq 1$, $P^{nt} \cdot s^n \subseteq a^nS_0$. Let $0 \neq c \in S$ satisfy $c \cdot S_0 \subseteq S$. Then, $P^{nt} \cdot (cs^n) \subseteq a^nS \subseteq \overline{a^nS}$, for all n.

If $cs^n \in \overline{a^nS}$ for all n, then $c \in a^nS'_Q$, for all n, since $s \notin Q$. But then $c \in \bigcap_{n\geq 1} a^nS'_Q = 0$, since S'_Q is a DVR. This is a contradiction. Thus $cs^n \notin \overline{a^nS}$, for some n, which implies $P \in \mathrm{Ass}\ (S/\overline{a^nS})$, which is what we want. □

[1]If $Q \neq PS[x] = P[x]$, then in the ring obtained by localizing at P and modding out P, the image of Q in $k(P)[x]$ is just a maximal ideal generated by an irreducible polynomial in $k(P)[x]$. Therefore, modding out this image gives an algebraic extension of $k(P)$.

Remark. Note that the last paragraph of the proof above shows that if S is a Noetherian domain and $0 \neq a \in S$, then $\bigcap_{n\geq 1} \overline{a^n S} = 0$. This is extended to arbitrary ideals below.

Corollary E3. *Let S be a Noetherian ring and $a \in S$ be a non-zerodivisor. Then $\bigcup_{n\geq 1} Ass\ (S/\overline{a^n S})$ is finite.*

Proof. Immediate from C3 and D3.

We need one more lemma before we can show that $\bigcup_{n\geq 1} \text{Ass}(R/\overline{I^n})$ is finite.

Lemma F3. *Let $I \subseteq R$ be an ideal and $\mathcal{R} := R[It, t^{-1}]$ denote the extended Rees aring of R with respect to I. Then for all $n \geq 1$:*

(i) $\overline{I^n} = \overline{t^{-n}\mathcal{R}} \cap R$.

(ii) The vth graded component of $\overline{t^{-n}\mathcal{R}}$ is $(I^v \cap \overline{I^{n+v}})t^v$, for all v.

Proof. For (i), take $a \in R$. Suppose $a \in \overline{I^n}$. Then there exists an equation of the form

$$a^s + i_n a^{s-1} + i_{2n} a^{s-2} + \cdots + i_{sn} = 0,$$

where each $i_{jn} \in I^n$. Multiply this equation by t^{sn} to get

$$(\frac{a}{t^{-n}})^s + i_n t^n (\frac{a}{t^{-n}})^{s-1} + i_{2n} t^{2n} (\frac{a}{t^{-n}})^{s-2} + \cdots + i_{sn} t^{sn} = 0. \quad (*)$$

This shows $\frac{a}{t^{-n}}$ is integral over $\mathcal{R}$, so $a \in \overline{t^{-n}\mathcal{R}}$.

Conversely, if $a \in \overline{t^{-n}\mathcal{R}} \cap R$, then $\frac{a}{t^{-n}}$ is integral over $\mathcal{R}$. By comparing terms of the same degree in an equation of integral dependence of $\frac{a}{t^{-n}}$ over $\mathcal{R}$, we may work backwards from an equation like (*) to show $a \in \overline{I^n}$. ☐

The proof of (ii) is almost the same. Suppose $ct^r \in \overline{t^{-n}\mathcal{R}}$. Then clearly $c \in I^v$. On the other hand, $\frac{ct^v}{t^{-n}} = ct^{v+n}$ is integral over $\mathcal{R}$. Thus, there exists an equation of the form

$$(ct^{v+n})^s + f_1(ct^{v+n})^{s-1} + \cdots + f_s = 0,$$

with $f_i \in \mathcal{R}$. Taking the coefficient of $t^{s(tn+v)}$ in this equation gives

$$c^s + j_1 c^{s-1} + \cdots + j_s = 0,$$

where each $j_i \in I^{i(v+n)}$. Thus, $c \in \overline{I^{n+v}}$, which gives what we want. The proof of the converse is similar, ☐

Theorem G3. *For any ideal $I \subseteq R$, $\bigcup_{n\geq 1} Ass(R/\overline{I^n})$ is finite.*

Proof. Suppose $P = (\overline{I^n} : c)$, for some $n \geq 1$ and $c \notin \overline{I^n}$. Localize R at P. By the previous lemma, $c \notin \overline{t^{-n}\mathcal{R}}$. Thus P consists of zero divisors on $\mathcal{R}/\overline{t^{-n}\mathcal{R}}$. It follows that $P\mathcal{R} \subseteq Q$, for some $Q \in \text{Ass}\ (\mathcal{R}/\overline{t^{-n}\mathcal{R}})$. Thus, $Q \cap R = P$. By Corollary $E3$, $\bigcup_{n\geq 1} \text{Ass}\ (\mathcal{R}/\overline{t^{-n}\mathcal{R}})$ is finite. This forces $\bigcup_{n\geq 1} \text{Ass}\ (R/\overline{I^n})$ to be finite, which is what we want. ☐

Remarks. (i) We denote the finite set of prime ideals in Theorem G3 by $\overline{A^*}(I)$. Note that $x \in R$ is a zerodivisor modulo $\overline{I^n}$ for some n if and only if $x \in P$, for some $P \in \overline{A^*}(I)$.

(ii) The proof of the Theorem G3 can be adapted to show the following: If $R \subseteq S$ are Noetherian rings, and $J \subseteq S$ is an ideal, then, if $P \in \text{Ass}\ R/(J \cap R)$, there exists $Q \in \text{Ass}\ S/J$ such that $Q \cap R = P$.

(iii) There is a stronger version of Theorem G3. Ratliff has shown that if height$(I) > 0$, then the sets $\text{Ass}\ R/\overline{I} \subseteq \text{Ass}R/\overline{I^2} \subseteq \cdots$ form an ascending chain. Since the union of these set is finite, this increasing chain of sets must stabilize and we have that there exists an n_0 such that $\bigcup_{n\geq 1} \text{Ass}\ R/\overline{I^n} = \text{Ass}\ R/\overline{I^{n_0}}$.

(iv) The stronger statement in (iii) is an integral closure analogue of a theorem due to M. Brodmann who showed that for all finitely generated R-modules M and ideals $I \subseteq R$, Ass $(M/I^n M)$ is stable for n sufficiently large. However, the sets Ass $(M/I^n M)$ need not be an increasing set of prime ideals. When $M = R$, we will denote this stable value $A^*(I)$. A theorem of Ratliff shows that $\overline{A^*}(I) \subseteq A^*(I)$.

Definition. A sequence of elements $x_1, \ldots, x_r \in R$ is said to be an *asymptotic sequence* if for each $1 \leq i \leq r$, x_i does not belong to any prime ideal in $\overline{A^*}((x_1, \ldots, x_{i-1})R)$. In other words, for all i, x_i is not a zerodivisor modulo $\overline{(x_1, \ldots, x_{i-1})^n R}$, for all n.

Remarks. (i) Asymptotic sequences in the form above were defined independently Ratliff and D. Katz. Earlier, Rees had defined the notion of an *asymptotic sequence over* I, for an ideal I contained in a local ring. His definition was to assume that x_i is not a zero divisor modulo $\overline{(I, x_i, \ldots, x_{i-1})^n R}$, for all n and all i. Rees used this concept to improve an earlier inequality of Burch that related the *analytic spread* of an ideal $I \subseteq R$ to a difference between the dimension of R and the the depths of the modules R/I^n.

(ii) Ratliff and DK studied properties of asymptotic sequences, discarding the ideal I. They independently proved (DK in his UT Austin PhD thesis) that a local ring is quasi-unmixed if and only if some (every) system of parameters forms an asymptotic sequence. This theorem will be our next goal. Using this result one can give a natural proof of Ratliff's theorem, once one knows a little about how the dimension formula is related to the universally catenary property.

(iii) A regular sequence is an asymptotic sequence, though this is not obvious from the definitions. However, this is clear in the case of a single element, because x is the first element in a regular sequence if and only if x is a non-zerodivisor, while x is is the first element in an asymptotic sequence if and only if height$(xR) = 1$, since $\overline{A^*}(0)$ is the set of minimal prime ideals of R.

Our immediate goal is to proof some basic properties about asymptotic sequences. The following theorem is due to Ratliff and DK.

Theorem H3. *Let* $(R, \mathfrak{m})$ *be a local ring and assume dim*$(R) > 0$*. Let* $x_1, \ldots, x_r \in R$ *be a sequence of elements. The following properties hold.*

(i) $x_1, \ldots, x_r$ form an asymptotic sequence if and only if they form an asymptotic sequence in $\widehat{R}$.
(ii) $x_1, \ldots, x_r$ form an asymptotic sequence if and only if their images in R/q form an asymptotic sequnce for all minimal primes $q \subseteq R$.
(iii) $x_1, \ldots, x_r$ form an asymptotic sequence if and only the images of $x_1, \ldots, x_r$ in $\widehat{R}/z$ generate an ideal of height r, for all minimal primes $z \subseteq \widehat{R}$.
(iv) Any permutation of an asymptotic sequence is an asymptotic sequence.
(v) The set of all maximal asymptotic sequences in R have the same length, which is equal to the minimum of $\dim(\widehat{R}/z)$, taken over all minimal primes $z \subseteq \widehat{R}$.

We will need a number of preliminary results in order to prove Theorem H3.

Proposition I3. *Let* R *be a Noetherian ring and* $I \subseteq R$ *and ideal. Let* $\mathcal{R}$ *denote the extended Rees ring of* R *with respect to* I*. Then* $P \in \overline{A^*}(I)$ *if and only if there exists* $Q \in \overline{A^*}(t^{-1}\mathcal{R})$ *such that* $Q \cap R = P$*.*

Proof. If $P \in \overline{A^*}(I)$, then the proof of Theorem G3 shows that there exists $Q \in \overline{A^*}(t^{-1}\mathcal{R})$ such that $Q \cap R = P$. Conversely, suppose there exists $Q \in \overline{A^*}(t^{-1}\mathcal{R})$ such that $Q \cap R = P$. Without lost of generality we may assume R is local at P. Since $\mathcal{R}$ is a graded ring, we can write $Q = (\overline{t^{-n}\mathcal{R}} : ct^v)$, for some $ct^v \in \mathcal{R}$. Since $ct^v \notin \overline{t^{-n}\mathcal{R}}$, Lemma F3 gives $c \notin \overline{I^{n+v}}$. Note that this implies that $n + v \geq 1$, since the degree j components of $\mathcal{R}$ equal R when $j \leq 0$. On the other hand, $P \subseteq Q$, so in $\mathcal{R}$ we have $P \cdot ct^n \subseteq \overline{t^{-n}\mathcal{R}}$, which by Lemma F3 gives $P \cdot c \subseteq \overline{I^{n+v}}$. Since P is maximal, we have $P \in \text{Ass } (R/\overline{I^{n+v}}) \subseteq \overline{A^*}(I)$, as required. ☐

Lemma J3. *Let* S *be a Noetherian ring and* $J \subseteq S$ *an ideal. Then* $\bigcap_{n\geq 1} \overline{J^n} = N_0$*, where* N_0 *is the intersection of the minimal primes* $q \subseteq R$ *satisfying* $J + q \neq S$*. In particular, if* J *is contained in the Jacobson radical of* S*, then* $\bigcap_{n\geq 1} \overline{J^n} = N$*, the nilradical of* S*.*

Proof. Let us first assume the result holds when S is an integral domain. Let $a \in \bigcap_{n\geq 1} \overline{J^n}$. If $q \subseteq S$ is a minimal prime such that $J + q \neq S$, then $a \in \bigcap_{n\geq 1} \overline{(J^n + q)/q} \equiv 0 \bmod q$ by the domain case, and thus a belongs to q. Conversely, if $a \in N_0$, let $q_1, \ldots, q_t$ denote the minimal primes of S for which $J + q_i = S$. Note, if there are no such primes, $a \in N$, which clearly belongs to $\overline{J^n}$, for all n. Now, $J^n + q_i = S$, for all n and all i. Fix n. For each i, we can write $1 + j_i = x_i$, with $j_i \in J^n$ and $x_i \in q_i$. Thus $b := a(1 + j_1) \cdots (1 + j_t)$ belongs to the nilradical of S. Therefore $b^c = 0$ for some $c \geq 1$. But this gives an equation of integral dependence of a on J^n, which shows $a \in \overline{J^n}$, for all n.

Now suppose S is a Noetherian domain and $J \subseteq S$ is a proper ideal. Let $\mathcal{R}$ denote the extended Rees ring of S with respect to J. By Lemma F3, $\overline{J^n} = \overline{t^{-n}\mathcal{R}} \cap S$, for all n. On the other hand, the remark following Proposition D3 shows $\bigcap_{n\geq 1} \overline{t^{-n}\mathcal{R}} = 0$, which completes the proof. ☐

Proposition K3. *Let S be a Noetherian ring, $J \subseteq S$ an ideal with height$(J) > 0$ and $q \subseteq S$ a minimal prime ideal. Suppose the prime ideal P is minimal over $J + q$. Then there exists $n \geq 1$ with the following property: $P \in Ass\ (R/L)$ for all ideals $J^n \subseteq L \subseteq \overline{J^n}$.*

Proof. Standard properties of localization show that if n satisfies the conclusion of the proposition for R_P, it also satisfies the conclusion for R. Thus, we may assume R is local at P. It follows that $P^s \subseteq J + q$ for some s. On the other hand, since q is a minimal prime, there exists $b \notin q$ such that $bq^c = 0$, for some $c \geq 1$. Since $b \notin q$, by the previous lemma, there exists n such that $b \notin \overline{J^n}$. Now $P^{sc} \subseteq J + q^c$, so $P^{scn} \subseteq J^n + q^{nc}$. Thus, $P^{scn} \cdot b \subseteq J^n \subseteq L$, for all L between J^n and $\overline{J^n}$. Since $b \notin \overline{J^n}$, P^{scn} consists of zero-divisors modulo L. Since P is maximal, we have $P \in \text{Ass}\ (S/L)$, as required. □

The next proposition due to Ratliff plays an important role in our story.

Proposition L3. *Let S be a Noetherian domain and $0 \neq a \in S$. Suppose S satisfies the dimension formula. If $P \in \overline{A^*}(aS)$, then height$(P) = 1$.*

Proof. By Proposition D3, there exists a height one prime $Q \subseteq S'$ such that $Q \cap S = P$. Let $Q, Q_2, \ldots, Q_r$ be the prime ideals in S' lying over P and take $u \in Q \backslash Q_2 \cup \cdots \cup Q_r$. Note that on the one hand, the Mori-Ngata theorem guarantees that there are just finitely many primes in S' lying over P, while on the other hand, since S' is integral over S, the primes lying over P are incomparable, so we may choose such a u. Set $Q_0 = Q \cap S[u]$. We claim height$(Q_0) = 1$. Suppose the claim holds. Note $Q_0 \cap R = P$. Then since $S[u]$ is integral over S and thus $k(Q_0)$ is algebraic over $k(P)$, the dimension formula applied to the extension $S[\subseteq S[u]$ shows that height$(P) = 1$, which is what we want.

For the claim, since $u \in Q_0$, the choice of u implies that Q is the only prime in S' lying over Q_0, since any such prime lies over P. If height$(Q_0) > 1$, we take take $Q' \subsetneq Q_0$. By the going up property, there are primes $Q_c \subseteq Q_d$ in S' such that $Q_c \cap S[u] = Q'$ and $Q_d \cap S[u] = Q_0$. But Q is the only prime in S' lying over Q_0, so $Q = Q_d$. But this contradicts height$(Q) = 1$. Thus, we must have height$(Q_0) = 1$ and the proof is complete. □

The proof of the next theorem requires the fact that a complete local domain satisfies the dimension formula. We will eventually prove this fact below.

Theorem M3. *Let $(R, \mathfrak{m})$ be a local ring and $I \subseteq R$ an ideal. Then for a prime ideal $P \subseteq R$, $P \in \overline{A^*}(I)$ if and only if there exists $Q \in \overline{A^*}(I\widehat{R})$ such that $Q \cap R = P$.*

Proof. We first note that for any ideal $J \subseteq R$, $\overline{J\widehat{R}} \cap R = \overline{J}$. If this holds, then the forward direction of the theorem follows from Remark (ii) following Theorem G3. Clearly $\overline{J} \subseteq \overline{J\widehat{R}} \cap R$. Conversely, suppose $a \in R$ satisfies an equation of the form

$$a^s + j_1 a^{s-1} + \cdots + j_s s,$$

where each $j_k \in J^k\widehat{R}$. Thus, $a^s \in (a^{s-1}J, a^{s-2}J^2, \ldots, J^s)\widehat{R} \cap R = (a^{s-1}J, a^{s-2}J^2, \ldots, J^s)R$, by faithful flatness. This last relation shows $a \in \overline{J}$.

For the converse, suppose $Q \in \overline{A^*}(I\widehat{R})$ and $Q \cap R = P$. Set $\widehat{\mathcal{R}} := \mathcal{R} \otimes \widehat{R}$, a faithfully flat extension of $\mathcal{R}$, and note that $\widehat{\mathcal{R}}$ is the extended Rees ring of $\widehat{R}$ with respect to $I\widehat{R}$. By proposition I3, there exists $Q_0 \in \text{Ass}\ (\widehat{\mathcal{R}}/\overline{t^{-n}\widehat{\mathcal{R}}})$, for some n, with $Q_0 \cap \widehat{R} = Q$. By Corollary C3, there exists a minimal prime $q \subseteq Q_0$ such that, if we write $\widehat{\mathcal{R}}_q$ for $\widehat{\mathcal{R}}/q$, $Q_0/q \in \text{Ass}(\overline{t^{-n}\widehat{\mathcal{R}}_q})$. Now, $\widehat{\mathcal{R}}_q$ is a finitely generated algebra over the complete local domain $\widehat{R}/(q \cap \widehat{R})$. Since $\widehat{R}/(q \cap \widehat{R})$ satisfies the dimension formula, any finitely generated algebra over it satisfies the dimension formula. Thus, $\widehat{\mathcal{R}}_q$ satisfies the dimension formula. Therefore, by Proposition L3, height$(Q_0/q) = 1$.

It follows that Q_0 is minimal over $t^{-1}\widehat{\mathcal{R}} + q$. Thus, by Proposition K3, there exists an $n \geq 1$ such that $Q_0 \in \text{Ass}\ (\widehat{\mathcal{R}}/L)$, for all L between $t^{-n}\widehat{\mathcal{R}}$ and $\overline{t^{-n}\widehat{\mathcal{R}}}$. Thus, if we take $L = (\overline{t^{-n}\mathcal{R}})\widehat{\mathcal{R}}$, we have $Q_0 \in \text{Ass}\ (\widehat{\mathcal{R}}/(\overline{t^{-n}\mathcal{R}})\widehat{\mathcal{R}})$. By faithful flatness $P_0 := Q_0 \cap \mathcal{R}$ be longs to $\text{Ass}\ (\mathcal{R}/\overline{t^{-n}\mathcal{R}}) \subseteq \overline{A^*}(t^{-1}\mathcal{R})$. By Proposition I3, $P_0 \cap R \in \overline{A^*}(I)$. Since $P = P_0 \cap R$, the proof is complete. □

The proof of the following corollary has a similar strategy to that of the previous theorem. However, we will use the theorem itself in the proof of the corollary. We will also use the following notation. If S is a ring and $q \subseteq S$, S_q will denote S/q. If $J \subseteq S$ is an ideal, we will use J_q to denote the image of J in S/q.

Corollary N3. Let $I \subseteq R$ be an ideal. For a prime ideal $P \subseteq R$, $P \in \overline{A^*}(I)$ if and only if there exists a minimal prime $q \subseteq R$ with $P_q \in \overline{A^*}(I_q)$.

Proof. The forward direction follows from Corollary C3. For the reverse direction, suppose $P_q \in \overline{A^*}(I_q)$ for some minimal prime $q \subseteq R$. Without loss of generality we may assume R is local at P. By the theorem above, we may lift P_q to a prime belonging to $\overline{A^*}(I_{q\widehat{R}})$, since $\widehat{R}_q$ is the completion of R_q. Since P is the maximal ideal of R, we get $P\widehat{R}_q \in \overline{A^*}(I\widehat{R}_q)$. By Corollary C3, there exists a minimal prime $\widehat{q} \subseteq \widehat{R}$ containing $q\widehat{R}$ such that $P_{\widehat{q}} \in \overline{A^*}(I_{\widehat{q}})$. If the conclusion of Corollary I3 holds when R is complete, then we have $P\widehat{R} \in \overline{A^*}(I\widehat{R})$, and thus $P \in \overline{A^*}(I)$, by Theorem M3. Thus, we may begin again assuming (R, P) is a complete local ring.

To continue, we need to use the extended Rees ring of R_q with respect to I_q. It is straightforward to check that this ring is just $\mathcal{R}_{q*}$, where $q* = qR[t, t^{-1}] \cap \mathcal{R}$, where $\mathcal{R}$ is the extended Rees ring of R with respect to I. Thus by Proposition I3 there exists a prime $Q' \in \overline{A^*}(t^{-1}\mathcal{R}_{q*})$ such that $Q' \cap R_q = P_q$. But $Q' = Q_{q*}$, for a prime $Q \subseteq \mathcal{R}$, and $q* \subseteq Q$. Since $\mathcal{R}_{q*}$ satisfies the dimension formula, Q_{q*} has height one, by Proposition L3. Therefore, Q is minimal over $t^{-1}\mathcal{R} + q*$. By Proposition K3, $Q \in \overline{A^*}(t^{-1}\mathcal{R})$. Thus, by Proposition I3, $P = Q \cap R$ belongs to $\overline{A^*}(I)$, as required. □

We need two more preliminary results before proving Theorem H3. The first will be presented as a remark.

Remark. We mentioned above that a complete local ring is catenary. Thus a complete local domain is a catenary domain. In such a ring, say $(R, \mathfrak{m})$, if I is a height r ideal generated by r elements $x_1, \ldots, x_r$, then for all $1 \leq t < r$, the ideal generated by $x_1, \ldots x_t$ has height t. To see this, since the sequence $x_1, \ldots, x_r$ can be completed to a system of parameters for R, $\dim R/(x_1, \ldots, x_t)R = d - t$, where $d := \dim(R)$. Let Q be a prime minimal over $(x_1, \ldots, x_t)R$ and suppose height$(Q) < t$. Since $\dim(R/Q) \leq d - t$, it follows that a saturated chain of primes from (0) to $\mathfrak{m}$ that passes through Q has length less that d. This contradicts that R is catenary. Therefore height$(Q) \geq t$. By Krull's principal ideal theorem height$(Q) \leq t$. Therefore, Q has height r, which shows $(x_1, \ldots, x_t)R$, has height t.

The following Proposition is due to Ratliff.

Proposition O3. *Let R be a Noetherian domain satisfying the dimension formula and $I \subseteq R$ an ideal. If $P \in \overline{A^*}(I)$, then height$(P) \leq \mu(I_P)$, where $\mu(I_P)$ denote the minimal number of elements generating I_P.*

Proof. We may assume R is local at P. Write $I = (a_1, \ldots, a_d)R$. Let $\mathcal{R}$ denote the extended Rees ring of R with respect to I. Note that $\mathcal{R}$ is generated as an R-algebra by $d + 1$ elements. Now, by Proposition I3, there exists $Q \in \overline{A^*}(t^{-1}\mathcal{R})$ such that $Q \cap R = P$. Since R satisfies the dimension formula, $\mathcal{R}$ does as well, so by Proposition L3 height$(Q) = 1$. We have

$$\text{height}(Q) + \text{tr.deg}_{k(P)}k(Q) = \text{height}(P) + 1.$$

Since $t^{-1} \in Q$, $\mathcal{R}/Q$ is generated as an algebra over R/P by no more than d elements. Thus, the transcendence degree of this extension is at most d. Using this in the displayed formula above shows height$(P) \leq d$, as required. □

We can now prove Theorem H3.

Proof of Theorem H3. Statement (i) follows immediately from Theorem M3. For example, suppose $x_1, \ldots, x_r$ is an asymptotic sequence. If they do not remain an asymptotic sequence in $\widehat{R}$, then for some $j \leq r$, there exists $Q \in \overline{A^*}((x_1, \ldots, x_{j-1})\widehat{R})$ with $x_j \in Q$. By Theorem M3, $P = Q \cap R$ belongs to $\overline{A^*}((x_1, \ldots, x_{j-1})R)$. Since $x_j \in P$, this is a contradiction. The converse is similar.

The proof of (ii) in Theorem H3 is similar to part (i), only one uses Corollary N3. Suppose $x_1, \ldots, x_r$ is an asymptotic sequence. Let $q \subseteq R$ be a minimal prime, and maintain the notation from Corollary N3. If the x_i do not remain an asymptotic sequence in R_q, then for some $j \leq r$, there exists $Q_q \in \overline{A^*}((x_1, \ldots, x_{j-1})R_q)$ with the image of x_j in R_q belonging to Q_q. Here $Q \subseteq R$ is a prime in R containing q. By Corollary N3, Q belongs to $\overline{A^*}((x_1, \ldots, x_{j-1})R)$. Since $x_j \in Q$, this is a contradiction. The converse is similar.

For (iii), we first note that by parts (i) and (ii), the given x_i form an asymptotic sequence if and only if their images in $\widehat{R}_z$ form an asymptotic sequence, for all minimal primes $z \subseteq \widehat{R}$. Thus, we must prove that if R is a complete local domain, then $x_1, \dots, x_r$ form an asymptotic sequence if and only if height$(x_1, \dots, x_r)R = r$, for all i. For this, suppose $x_1, \dots, x_r$ is an asymptotic sequence. Since each x_i is chosen to avoid the primes in $\overline{A^*}((x_i, \dots, x_{i-1})R)$, each x_i avoids the primes minimal over $(x_1, \dots, x_{i-1})R$. Therefore the ideals $(x_1, \dots, x_i)R$ all have height i. Conversely, suppose $x_1, \dots, x_r$ generate an ideal having height r. Then, by the remark above, the ideal generated by each $x_1, \dots x_t$ has height t, for all $1 \leq t < r$. Suppose $x_1, \dots, x_r$ do not form an asymptotic sequence. Then for some j, $x_j \in P$, for some $P \in \overline{A^*}((x_1, \dots, x_{j-1})R)$. By Proposition O3, height$(P) \leq j-1$. On the other hand, $(x_1, \dots, x_{j-1})R \subseteq P$, so height$(P) \geq j-1$, and therefore height$(P) = j-1$. Since $x_j \in P$, this contradicts the assumption on the x_i. Thus, $x_1, \dots, x_r$ form an asymptotic sequence.

Part (iv) Follows immediately from part (iii).

For part (v) we use the obvious terminology: We say that $x_1, \dots, x_s$ form a *maximal* asymptotic sequence if they form an asymptotic sequence and there does not exist $y \in R$ such that $x_1, \dots, x_s, y$ is an asymptotic sequence. The second condition is equivalent to requiring $\mathfrak{m} \in \overline{A^*}((x_1, \dots, x_s)R)$. Set $\delta(R)$ to be the minimum of $\dim(\widehat{R}/z)$, taken over all minimal primes $z \subseteq \widehat{R}$. By part (iii), the length of any asymptotic sequence is less than or equal to $\delta(R)$, including the length of a maximal asymptotic sequence. Now suppose $x_1, \dots, x_s$ is a maximal asymptotic sequence. Then $\mathfrak{m} \in \overline{A^*}((x_1, \dots, x_s)R)$. By parts (i) and (ii) above, there exists a minimal prime $z \subseteq \widehat{R}$ with $\mathfrak{m}\widehat{R}_z \in \overline{A^*}((x_1, \dots, x_s)\widehat{R}_z)$. By Proposition O3, $\dim(R_z) \leq s$. Thus, $\delta(R) \leq s$, which shows that all maximal asymptotic sequences in R have length $\delta(R)$. □

We can now state and prove the characterization of quasi-unmixed local rings.

Theorem P3. *Let $(R, \mathfrak{m})$ be a local ring. The following statements are equivalent.*

(i) R is quasi-unmixed.
(ii) Every system of parameters forms an asymptotic sequence.
(iii) Some system of parameters forms an asymptotic sequence.

Proof. We let $\delta(R)$ have the same meaning as above. Set $d := \dim(R)$. If R is quasi-unmixed, then $\delta(R) = d$. Let $x_1, \dots, x_d$ be a system of parameters and let I denote the ideal they generate. Then I is $\mathfrak{m}$-primary. It follows that the image of I in each $\widehat{R}_z$ is $\mathfrak{m}\widehat{R}_z$-primary for all minimal primes $z \subseteq \widehat{R}$. Each $\widehat{R}_z$ has dimension d, therefore the images of $x_1, \dots, x_d$ in each $\widehat{R}_z$ form a system of parameters and thus generate an ideal of height d. By Theorem H3, $x_1, \dots, x_d$ is an asymptotic sequence. So, (i) implies (ii). Clearly (ii) implies (iii). Finally, if some system of parameters forms an asymptotic sequence, this is clearly a maximal asymptotic sequence. The length of such equals $\delta(R)$ by Theorem H3. Thus $\delta(R) = \dim(R)$, and therefore R is quasi-unmixed. □

As a corollary, we can prove one component of Ratliff's Theorem.

Corollary Q3. *Let $(R, \mathfrak{m})$ be a local domain. If R satisfies the dimension formula, then R is quasi-unmixed.*

Proof. By the previous theorem it suffices to show that R has a system of parameters forming an asymptotic sequence. Suppose $x_1, \dots, x_r$ is a maximal asymptotic sequence. Then, $\mathfrak{m} \in \overline{A^*}(x_1, \dots, x_r)R$. On the other hand, by Proposition O3, height$(\mathfrak{m}) \leq r$. Since $r \leq$ height$(\mathfrak{m})$, we must have $r =$ height$(\mathfrak{m}) = \dim(R)$. This implies that $x_1, \dots, x_r$ is a system of parameters, and thus R is quasi-unmixed. □

We now want to work directly towards the other parts of Ratliff's theorem. We start with two elementary observations, one each concerning the dimension formula and the catenary property.

Observations 1. For Noetherian domains $A \subseteq B$ such that B is a finitely generated A-algebra, if A satisfies the dimension formula, then B satisfies the dimension formula. To see this, let C be a finitely generated B algebra. Then C is also a finitely generated A algebra. Let $Q \subseteq C$ be a prime ideal and set $P := Q \cap B$ and $P_0 := Q \cap A$. Then since A satisfies the dimension formula:

$$\text{height}(Q) + \text{tr.deg}_{k(P_0)} k(Q) = \text{height}(P_0) + \text{tr.deg}_A C,$$

and

$$\text{height}(P) + \text{tr.deg}_{k(P_0)} k(P) = \text{height}(P_0) + \text{tr.deg}_A B.$$

Solving each equation for height(P_0 and setting them equal to each other gives:

$$\text{height}(Q) + \text{tr.deg}_{k(P_0)}k(Q) - \text{tr.deg}_A C = \text{height}(P) + \text{tr.deg}_{k(P_0)}k(P) - \text{tr.deg}_A B.$$

Rewriting, we get:

$$\text{height}(Q) + \text{tr.deg}_{k(P_0)}k(Q) - \text{tr.deg}_{k(P_0)}k(P) = \text{height}(P) + \text{tr.deg}_A C - \text{tr.deg}_A B.$$

Additivity of transcendence degree gives:

$$\text{height}(Q) + \text{tr.deg}_{k(P)}k(Q) = \text{height}(P) + \text{tr.deg}_B C,$$

which is what we want.

Observation 2. A Noetherian ring S is catenary if and only if for every pair of prime ideals $P \subseteq Q$, $\text{height}(Q) = \text{height}(P) + \text{height}(Q/P)$. To see this, suppose the height condition holds. Let $P \subseteq Q$ be prime ideals. To see that all saturated chains of primes between P and Q have the same length, we may mod out P and localize at Q. Note that these operations preserve the height condition. Thus, we have to show that if the height condition holds, all maximal chains of primes in a local domain $(R, \mathfrak{m})$ have the same length, namely, $\dim(R)$. Let $(0) \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_s = \mathfrak{m}$ be a maximal chain of length s. Clearly $\text{height}(Q_1) = 1$. By the height condition $\text{height}(Q_2) = \text{height}(Q_2/Q_1) + \text{height}(Q_1) = 1 + 1 = 2$, since, by assumption, there are no primes between Q_1 and Q_2. Continuing in this fashion, we see $\text{height}(Q_i) = i$, for all i. Thus, $s = \text{height}(Q_s) = \text{height}(\mathfrak{m}) = \dim(R)$, which is what we wanted to prove.

We can now state and prove a second implication in Ratliff's Theorem.

Proposition R3. *Let R be a universally catenary Noetherian domain. Then R satisfies the dimension formula.*

Proof. By the observation above, we just have to prove the following statement. If T is a Noetherian domain, and $T = R[x]$, for some $x \in T$, then the dimension formula holds between R and T. If x is algebraically independent over R, then we have verified the dimension formula in this case in Remark (iii) following the definition of the dimension formula. Suppose x is algebraic over R. Let A denote the polynomial ring in one variable over R set K to be the kernel of the natural homomorphism from A to T. Take a prime ideal $Q \subseteq T$ and set $P := Q \cap R$. Since $\text{tr.deg}_R T = 0$, we must show

$$\text{height}(Q) + \text{tr.deg}_{k(P)}k(Q) = \text{height}(P).$$

Let Q_0 denote the preimage of Q in A, so that $Q = Q_0/K$. Since A is catenary,

$$\text{height}(Q_0) = \text{height}(Q_0/K) + \text{height}(K) = \text{height}(Q) + 1. \quad (*)$$

Since the dimension formula holds between A and R we have

$$\text{height}(Q_0) + \text{tr.deg}_{k(P)}k(Q_0) = \text{height}(P) + \text{tr.deg}_R A = \text{height}(P) + 1.$$

Using (*) in this last equation we have

$$\text{height}(Q) + 1 + \text{tr.deg}_{k(P)}k(Q_0) = \text{height}(P) + 1. \quad (**)$$

But $A/Q_0 = T/Q$, so $\text{tr.deg}_{k(P)}k(Q_0) = \text{tr.deg}_{k(P)}k(Q)$. Substituting this into (**) and cancelling 1 yields $\text{height}(Q) + \text{tr.deg}_{k(P)}k(Q) = \text{height}(P)$, which is what we want. □

Here is a result of independent interest that plays a key role in our analysis.

Proposition S3. *Let S be a a Cohen-Macaulay ring. Then S is catenary.*

Proof. We just have to check the height condition in the observation above. Let $P \subseteq Q$ be primes. We may assume that S is local at Q. Suppose P has height h and set $d =: \dim(S)$. Take $\underline{x} = x_1, \ldots, x_h$ a maximal regular sequence from P. Then

$$\dim(S) - \text{height}(P) = d - h = \text{depth}(S/(\underline{x})) \leq \dim(S/P),$$

the latter inequality holds since P is an associated prime of the S-module $S/(\underline{x})$. On the other hand, $\dim(S/P) + \text{height}(P) \leq \dim(S)$ always holds in a local ring, and thus, $\dim(S) = \text{height}(P) + \dim(S/P)$, which is what we want. □

We have used the following proposition a few times already. Time now for its proof.

Proposition T3. *Let $(R, \mathfrak{m})$ be a complete local domain. Then R is universally catenary and satisfies the dimension formula.*

Proof. We use the fact that a homomorphic image of a catenary ring is catenary. This follows immediately from the definition and standard facts about primes in homomorphic images. To see that R is universally catenary, it suffices to show that a polynomial ring in finitely many variables over R is catenary. By Cohen's Structure Theorem, R is the homomorphic image of a regular local ring S. Hence any polynomial ring B over R is a homomorphic image of a polynomial ring A over S. Since S is Cohen-Macaulay, A is Cohen-Macaulay, and therefore catenary. Thus, B is catenary, which shows R is universally catenary.

The second statement is now immediate from Proposition R3, □

Remark. Since the catenary property does not require the ring in question to be an integral domain, the proof above shows that a complete local ring is catenary.

We are closing in on the last step in Ratliff's Theorem, namely that a quasi-unmixed local domain is universally catenary. The proof of the following Proposition is greatly facilitated by the use of asymptotic sequences.

Proposition U3. *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring. Leet $P \subseteq R$ be a prime ideal.*

(i) $\dim(R/P) + \text{height}(P) = \dim(R)$.
(ii) R/P is quasi-unmixed.
(iii) R_P is quasi-unmixed.
(iv) R is catenary.

Proof. Let $x_1, \ldots, x_r$ be an asymptotic sequence of maximal length from P. Then, there exists $P_0 \supseteq P$ with $P_0 \in \overline{A^*}((x_1, \ldots, x_r)R)$. By Theorem M3, there exists $Q \subseteq \widehat{R}$ with $Q \in \overline{A^*}((x_1, \ldots, x_r)\widehat{R})$ with $Q \cap R = P_0$. Moreover, there exists $z \subseteq \widehat{R}$, a minimal prime, so that $Q_z \in \overline{A^*}((x_1, \ldots, x_r)\widehat{R}_z)$. On the one hand, by Proposition O3, $\text{height}(Q_z) \leq r$, since R_z satisfies the dimension formula. On the other hand, by Theorem H3 (iii), $\text{height}(Q_z) \geq r$. Thus, $\text{height}(Q)_z = r$. Since R is catenary,

$$r = \text{height}(Q_z) = \dim(\widehat{R}/z) - \dim(\widehat{R}/Q) = \dim(\widehat{R}) - \dim(\widehat{R}/Q),$$

since R is quasi-unmixed. Therefore,

$$\dim(R/P) = \dim(\widehat{R}/P\widehat{R}) \geq \dim(\widehat{R}/Q) = \dim(\widehat{R}) - r \geq \dim(R) - \text{height}(P).$$

Thus $\dim(R) + \text{height}(P) \geq \dim(R)$. Since $\dim(R/P) + \text{height}(P) \leq \dim(R)$ always holds, (i) follows. Moreover, this shows $r = \text{height}(P)$ and $\dim(\widehat{R}/Q) = \dim(R/P)$. In addition, since $\text{height}(P_0) \leq r$, by Proposition O3, $P_0 = P$.

For (ii) Let P and $x_1, \ldots, x_r \in P$ be as in (i). Then P is minimal over $(x_1, \ldots, x_r)R$. Now assume Q is minimal over $P\widehat{R}$. Then Q is minimal over $(x_1, \ldots, x_r)\widehat{R}$ and thus belongs to $\overline{A^*}((x_1, \ldots, x_r)\widehat{R})$. By what we have shown in (i), it follows that $\dim(\widehat{R}/Q) = \dim(R) - r = \dim(R/P)$, so R/P is quasi-unmixed.

Now, since $r = \text{height}(P)$, upon localizing R at P, $x_1, \ldots, x_r$ becomes an asymptotic sequence of length $\dim(R_P)$, so R_P is quasi-unmixed by Theorem P3. This gives (ii).

Finally, suppose R is quasi-unmixed. Take $P \subseteq Q$ primes ideals. We have to check the height condition in Observation 2 above. We may localize R at Q. But then R_Q is quasi-unmixed by (iii) and by part (i), the the required height condition holds. □

Our last step requires us to show that if R is a quasi-umixed local ring and T is a polynomial ring in finitely many variables over R, then T is locally quasi-unmixed. We start with a lemma.

Lemma V3. *Let S be a Noetherian ring, $J \subseteq S$ an ideal and $R[x]$ the polynomial ring in one variable over R. Then $\overline{J[x]} = \overline{J}[x]$.*

Proof. Write $\mathcal{R}$ for the extended Rees algebra of R with respect to J and note that $\mathcal{R}[x]$ is the extended Rees algebra of $R[x]$ with respect to $J[x]$. Suppose $u(x) = u_n x^n + u_{n-1}x^{n-1} + \cdots + u_0$ belongs to $\overline{J[x]}$. By Lemma F3, $u(x) \in \overline{t^{-1}\mathcal{R}[x]}$. Thus, $\frac{u(x)}{t^{-1}}$ is integral over $\mathcal{R}[x]$. Therefore, each $\frac{u_i}{t^{-1}}$ is integral over $\mathcal{R}$ for each i, and thus $u_i \in \overline{t^{-1}\mathcal{R}}$, for each i. By Lemma F3, $u_i \in \overline{J}$ for all i, and hence, $u(x) \in \overline{J}[x]$, as required. □

Corollary W3. *Let $R[x]$ be the polynomial ring in one variable over R. If $x_1, \dots, x_r$ is an asymptotic sequence in R, then $x_1, \dots, x_r$ is an asymptotic sequence in $R[x]$.*

Proof. For any ideal $I \subseteq R$, $Q \in \text{Ass}(R[x]/I[x])$ if and only if there exists $P \in \text{Ass}(R/I)$, with $Q = P[x]$. Thus, in light of the previous lemma, for any ideal $J \subseteq R$, $Q \in \overline{A^*}(J[x])$ if and only if $Q = P$, for some $P \in \overline{A^*}(J)$. The corollary now follows from the definition of asymptotic sequence. □

Theorem X3. *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring and $T := R[x_1, \dots, x_n]$ be the polynomial ring in n variables over R. Then for any prime ideal $Q \subseteq T$, T_Q is quasi-unmixed.*

Proof. We induct on n. Suppose $n = 1$ and $Q \subseteq T$. Write $P := Q \cap R$. Then T_P is the polynomial ring in one variable over R_P and $(T_P)_Q = T_Q$. By Proposition U3, R_P is quasi-unmixed. Thus, we may begin again assuming $Q \cap R = \mathfrak{m}$. Since R is quasi-unmixed, by Theorem P3, every system of parameters forms an asymptotic sequence. Let $x_1, \dots, x_d \in R$ be a system of parameters, where $d := \dim(R)$. Then: (i) By Corollary W3, $x_1, \dots, x_d$ remain an asymptotic sequence in $R[x]$ and (ii) $\overline{A^*}((x_1, \dots, x_d)R) = \mathfrak{m}$. Moreover, the proof of Corollary W3 shows $\overline{A^*}((x_1, \dots, x_d)[x]) = \mathfrak{m}[x]$.

Now, since $Q \cap R = \mathfrak{m}$, we have two cases. If $Q = \mathfrak{m}[x]$, then $\dim(T_Q) = d$, and $x_1, \dots, x_d \in Q$ is a system of parameters in T_Q forming an asymptotic sequence. Thus, by Theorem P3, T_Q is quasi-unmixed. If $Q \neq \mathfrak{m}[x]$, then $Q = (\mathfrak{m}, f(x))T$, where $f(x)$ is a monic polynomial which is irreducible over $R/\mathfrak{m}$. Thus $f(x) \notin \mathfrak{m}[x] = \overline{A^*}((x_1, \dots, x_d)[x])$. Therefore, $x_1, \dots, x_d, f(x)$ form an asymptotic sequence in T, and also in T_Q. Since $\dim(T_Q) = d+1$, these elements are also a system of parameters in T_Q. Thus, T_Q is quasi-unmixed, by Theorem P3.

Now suppose $n > 1$. Let $Q \subseteq T$ be a prime ideal and set $P := Q \cap S$, where $S = R[x_1, \dots, x_{n-1}]$. By induction, S_P is quasi-unmixed. Since T_P is the polynomial ring in one variable over S_P, the $n = 1$ case gives that $(T_P)_Q = T_Q$ is quasi-unmixed, and the proof is complete. □

Here is the last component of Ratliff's Theorem.

Corollary Y3. *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring. Then R is universally catenary.*

Proof. It is enough to show that if T is a polynomial ring in finitely many variables over R, then T is catenary. For this, it suffices to show that T_Q is catenary for every prime $Q \subseteq T$. By Theorem $X3$, T_Q is quasi-unmixed and by Proposition U3, T_Q is catenary. □

For the sake of completeness, we put things all together.

Theorem (Ratliff). *Let $(R, \mathfrak{m})$ be a local integral domain. The following statements are equivalent.*

(i) R is quasi-unmixed.
(ii) R is universally catenary.
(iii) R satisfies the dimension formula.

Proof (i) implies (ii) by Corollary Y3. (ii) implies (iii) by Proposition R3. (iii) implies (i) by Corollary Q3. □

Here are two applications of the main results of this chapter.

Corollary Z3. *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring. Then:*

(i) For any finitely generated R-algebra S and primes $q \subseteq Q \subseteq S$, $(S/q)_Q$ is quasi-unmixed.
(ii) $R[[x]]$, the formal power series ring over R, is quasi-unmixed.

Proof. There are a couple of ways to see (i), given everything we have done. For example, by Ratliff's theorem, R is universally catenary, so S is universally catenary, and hence $(S/q)_Q$ is universally catenary, and therefore quasi-unmixed. Alternately, one can use the fact that if R is quasi-unmixed, then so is R/P for any prime $P \subseteq R$.

For (ii) one follows the ideas in the proof of Theorem X3 to show that if $a_1, \dots, a_d$ is a system of parameters in R forming an asymptotic sequence, then $a_1, \dots, a_d, x$ is a system of parameters forming an asymptotic sequence in $R[[x]]$. □

We close this section with an amusing observation about height one primes in the integral closure of a local domain.

Observation. *Let $(R, \mathfrak{m})$ be a local domain. Then there are only finitely many height one primes $Q \subseteq R'$ such that height$(Q \cap R) > 1$.*

Proof. Suppose $\{Q_n\}_n$ is an infinite set of height one primes in R' with height$(Q_n \cap R) > 1$, for all n. For each Q_i, we can take a non-zero element $a_i \in Q_i$. By Proposition D3, $Q_i \in \overline{A^*}((a_i R))$. By Theorem H3, Q_i lifts to a prime $P_i \in \overline{A^*}((a_i \widehat{R}))$. Each of these in turn have the property that $(P_i)_z \in \overline{A^*}((a_i \widehat{R}_z))$, for some minimal primes $z \in \widehat{R}$. Proposition L3 gives height$(P_i/z) = 1$. We are now in the following situation: we have infinitely many prime $P_i \subseteq \widehat{R}$ with height greater than one (since each Q_i has height greater than one), and each P_i has height one modulo some minimal prime. This cannot happen. Otherwise: since there are finitely many minimal primes in $\widehat{R}$, there must exist two minimal primes z_1, z_2 and infinitely many P_j such that height$(P_j/z_1) > 1$ and height$(P_j/z_2) = 1$, for all j. Take $b \in z_1 \backslash z_2$. Then the non-zero principal ideal $b\widehat{R}_{z_2}$ is contained in infinitely many height one primes, which is a contradiction, since an ideal in a Noetherian ring has only finitely many minimal primes. □

4. Fibers of Ring Maps and the Definition of an Excellent Local Ring

In this section we present a few definitions and properties of the fiber of a ring map as they relate to the definition of an excellent local ring. We do just enough to give the flavor of how things proceed when trying to prove basic properties about the remaining ingredients in the definition of an excellent local rings. Many of the important results are quite technical and require the development of a number of auxiliary ideas, so that we will not have time to go very far in this direction. We begin with some definitions. All rings are assumed to be Noetherian, unless stated otherwise.

Definitions. Let $\phi : R \to S$ be a ring homomorphism. Tensor products are taken over R.

(i) For $p \subseteq R$ a prime ideal, the *fiber of ϕ over p* is the $k(p)$-algebra $k(p) \otimes S$. Note that since $k(p)$ is just the ring R_U/p_U where $U = R \backslash p$, the fiber over p is just S_U/pS_U.
(ii) When R is local and $S = \widehat{R}$, the fibers of ϕ are called the *formal fibers* of R.
(iii) If R is local with residue field k, $k \otimes S$ is the *closed fiber* of the map ϕ.
(iv) If R is an integral domain, and $p = (0)$, so that $k(p) = K$, the quotient field of R, then $K \otimes S$ is called the *generic fiber* of ϕ.

Remark. The description of $k(p) \otimes S$ in (i) shows that the prime ideals in the fiber of ϕ over p correspond to the prime ideals in S contracting to p. In fact, as topological spaces, one can show that Spec$(k(p) \otimes S)$ is homeomorphic to the set of primes $P \in \text{Spec}(S)$ with $P \cap R = p$.

Examples. (i) Suppose R is a Noetherian ring and $S = R[x_1, \ldots, x_n]$ is the polynomial ring in n variables over R. Take $p \subseteq R$ any prime ideal. Then the fiber of the inclusion map $R \subseteq S$ is just $k(p)[x_1, \ldots, x_n]$. Thus, the fibers of this map look essentially the same, except the coefficient fields $k(p)$ can differ. However, the dimension of each fiber is the same, namely n. S is certainly faithfully flat over R, but this alone is not enough to insure that the fibers of the inclusion map all have the same dimension.

(ii) Let k be a field, and $R = \mathbb{Q}[y, z]$ be the polynomial ring in two variables over $\mathbb{Q}$. Then the formal power series ring $R[[x]]$ is faithfully flat over R. We now show that the fibers of the inclusion map $R \subseteq R[[x]]$ can have different dimensions. We need to use the following fact: There exist two power series $f(x), g(x) \in \mathbb{Q}[[x]]$ that are algebraically independent over $\mathbb{Q}$. In fact, the quotient field $\mathbb{Q}((x))$ of $\mathbb{Q}[[x]]$ has infinite transcendence degree over $\mathbb{Q}$. To see this, suppose the transcendence degree of $\mathbb{Q}((x))$ over $\mathbb{Q}$ were finite. Then we could find $f_1, \ldots, f_d \in \mathbb{Q}((x))$ such that $\mathbb{Q}((x))$ is algebraic over $K := \mathbb{Q}(f_1, \ldots, f_d)$. But K is a countable field and since an algebraic extension of a countable field is countable, that would imply that $\mathbb{Q}((x))$ is countable. But $\mathbb{Q}[[x]]$ is clearly uncountable. Thus, $\mathbb{Q}((x))$ has infinite (uncountable!) transcendence degree over $\mathbb{Q}$, so we may choose $f(x), g(x)$ as above.

Now define a ring homomorphism $\alpha : R[[x]] \to \mathbb{Q}[[x]]$ by sending sending $\mathbb{Q}$ to itself, y to $f(x)$, z to $g(x)$ and x to itself. Note that this ring map exploits the fact that R is isomorphic to $\mathbb{Q}[f(x), g(x)]$. Let P be the kernel of α. Then $y - f(x)$ and $z - g(x) \in P$. This forces P to have height 2. But $P \cap R = (0)$, which shows that the generic fiber of the inclusion $R \subseteq R[[x]]$ has dimension two. On the other hand, if we set $\mathfrak{m} = (y, z)R$, then $R[[x]]/\mathfrak{m}R[[x]] = \mathbb{Q}[[x]]$ is one dimensional, so the fiber over $\mathfrak{m}$ has dimension one. □

We begin with the following proposition, which gives some information about fibers of a ring homomorphism.

Proposition A4. *Let $\phi : R \to S$ be a ring homomorphism. For $P \subseteq S$, set $p := P \cap S$.* Then

$$\text{height}(P) \leq \text{height}(p) + \dim(k(p) \otimes S).$$

Equality holds if the going down property holds between R and S.

Proof. By localizing at P, R be comes local at p, and neither the heights or dimensions in question change. So we may assume that R is local at p and S is local at P. Set $d := \dim(R)$ and $t := \dim(S/pS)$. Let $x_1, \ldots, x_d \in R$ be a system of parameters and $y_1, \ldots, y_t \in S$ be such that their images in S/pS form a system of parameters. Then $P^c \subseteq \underline{y}S + pS$ and $p^d \subseteq \underline{x}R$, for some $c, d \geq 1$. Then $P^{c+d} \subseteq (\underline{x} + \underline{y})S$. This shows $\dim(S) \leq d + t$, which gives the first statement.

Now suppose the going down property holds between R and S. Let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_t = P$ be a saturated chain of primes containing pS. Note $P_0 \cap R = p$. Now let $p_0 \subsetneq \cdots \subsetneq p_d = p$ be a saturated chain in R. Then, by the going down property, in S, there is a chain of primes $P_0 = Q_d \supsetneq \cdots \supsetneq Q_0$ with $Q_i = p_i$, for all i. This gives a chain of primes of length $d + t$ in S. Thus, $\dim(S) = d + t$, which is what we want. ☐

Proposition B4. *Let $\phi : R \to S$ be a ring homomorphism so that S is faithfully flat over R. Then:*

(i) ϕ is injective.
(ii) The going down and lying over properties hold between R and S. In particular, equality holds in Proposition A4.
(iii) $\text{height}(I) = \text{height}(IS)$, for all ideals $I \subseteq R$.

Proof. For (i) suppose $a \in R$ is non-zero. We have an exact sequence $0 \to aR \to R$. If we tensor with S (via ϕ), the sequence $0 \to (aR) \otimes S \to R \otimes S = S$ stays exact, by the flatness of S over R. The image of $a \otimes 1_S$ under this map is just $\phi(a)$. If $\phi(a) = 0$, then $(aR) \otimes S = 0$, which contradicts the faithfully flat property.

For (ii), let $p \subseteq R$ be a prime ideal. Then the fiber $k(p) \otimes S$ is non-zero. Thus, $\text{Spec}(k(p) \otimes S)$ is non-empty, so by our comments above, there exists a prime $P \subseteq S$ with $P \cap R = p$. In other words, the lying over property holds. Now suppose P_2 is a prime ideal in S, and set $p_2 := P_2 \cap R$ and suppose we have a prime $p_1 \subsetneq p_2$. Localizing at P_2 preserves flatness (by transitivity of flatness), so we may assume S is local at P_2. Since p_2 is the only maximal ideal of R and $p_2 S \neq S$, the extension is also faithfully flat. By lying over, there is a prime $P_1 \subseteq S$ with $P_1 \cap R = p_1$. Since S is local at P_2, $P_1 \subseteq P_2$, so the going down property holds.

For part (iii), let $p \subseteq R$ be a prime ideal and take $P \subseteq S$ a prime minimal over pS. Again, we may localize at P and assume that S is local at P and faithfully flat over R. By part (ii), the going down property holds, so by Proposition A,

$$\text{height}(P) = \text{height}(p) + \dim(k(p) \otimes S) = \text{height}(p) + \dim(S/pS) = \text{height}(p) + 0 = \text{height}(p).$$

This argument shows $\text{height}(I) = \text{height}(IS)$. Indeed, if p is minimal over I having the same height as I, then the above shows $\text{height}(IS) \leq \text{height}(I)$. On the other hand, starting with P minimal over IS, P is minimal over pS, for $p = P \cap S$, so the argument shows $\text{height}(IS) \leq \text{height}(I)$, and therefore, $\text{height}(I) = \text{height}(IS)$. ☐

Here is a proposition that sheds some light on the dimension of fibers. Note that, in general, the going up property does not holds between R and a polynomial ring or a power series over R. Part (ii) of the example above shows that the going up property fails for power series rings, and that part (i) of the proposition below can fail in a faithfully flat extension, while if R is a DVR with uniformizing parameter π, going up fails for the extension $R \subseteq R[x]$, even though the fibers all have the same dimension. To see this, note that $(\pi x - 1)R[x]$ is a maximal ideal in the polynomial ring contracting back to zero. If we take the chain $(0) \subseteq (\pi)$ we cannot lift it to a chain in $R[x]$ starting with $(px - 1)$, since the latter is a maximal ideal.

Proposition C4. Let $\phi : R \to S$ be a ring homorphism between Noetherian rings. Let $q \subseteq p \subseteq R$ be prime ideals.

(i) If the going up holds, then $\dim(k(q) \otimes S) \leq \dim(k(p) \otimes S)$.
(ii) If going down holds, $\dim(k(p) \otimes S) \leq \dim(k(q) \otimes S)$.

Proof. For (i), let $r := \dim(k(q) \otimes S)$. Then there exists a chain of distinct primes $Q_0 \subseteq \cdots \subseteq Q_r$ in S such that for each i, $Q_i \cap R = q$. Suppose $s := \text{height}(p/q)$. Then in R, there exists a chain of distinct primes $q = p_0 \subseteq \cdots \subseteq p_s = p$. Since $p_0 = q$ by the going up property, we can lift this chain in R to a chain to $Q_r \subseteq \cdots \subseteq Q_{r+s}$, where each $Q_{r+j} \cap R = p_j$. Applying Proposition B4 to the induced homomorphism $R/q \to S/qS$. we have:

$$r + s \leq \text{height}(Q_{r+s}/qS) \leq \text{height}(p/q) + \dim(k(p/q) \otimes S/qS),$$

so $r \leq \dim(k(p/q) \otimes S/qS) = \dim(k(p) \otimes S)$, which is what we want.

For (ii), we first note that if the conclusion of part (ii) holds when $\text{height}(p/q) = 1$ then it holds in general. For if $q \subseteq p \subseteq p'$ with $\text{height}(p/q) = 1 = \text{height}(p'/p)$, then the longest chain of primes in S lying over p' is less than or equal to the longest chain of primes in S lying over p, which is less than or equal to the longest chain of primes in S lying over q, by the height one case. Iterating this shows we may assume $\text{height}(p/q) = 1$.

Set $r := \dim(k(p) \otimes S)$. If $r = 0$, there is nothing to prove. Now, suppose $r > 0$ and let $P_0 \subseteq \cdots \subseteq P_r$ be a chain of distinct primes in S with $P_j \cap R = p$, for all j. We need to find a chain of distinct primes $Q_0 \subseteq \cdots \subseteq Q_r$ in S, so that $Q_j \cap R = q$, for all j. For this we will use the following fact: If T is a Noetherian domain and C is a prime ideal in T having height greater than one, then C contains infinitely many height one primes of T.

Now, by going down, there exist $Q_0 \subsetneq P_0$ such that $Q_0 \cap R = q$. Take $x \in p \backslash q$. We apply the fact above to $T := S/Q_0$ and its prime P_1/Q_0, which has height greater than one. The fact above implies that there exists a height one prime contained in P_1/Q_0 not containing the image of x, since the image of x in T is contained in only finitely many height one primes This prime corresponds to a prime Q_1 in S containing Q_0, properly contained in P_1. Since x is not in Q_1, we can't have $Q_1 \cap R = p$ and since there are no primes in R between q and p, we must have $Q_1 \cap R = q$. We can now apply the same process in S/Q_1 to the prime P_2/Q_1 which has height greater one. There is a height one prime in S/Q_1 contained in P_2/Q_1 not containing the image of x. As before, this corresponds to a prime Q_2 properly containing Q_1, which satisfies $Q_2 \cap R = q$. Continuing in this fashion, we can create a chain of primes of length r in S where each element of the chain contracts to q. This complete the proof of the proposition. □

The next theorem shows how some familiar properties transfer between a Noetherian ring and a faithfully extension. The fibers of the ring map play a key role. But we first recall some definitions.

Remark Let R be a Noetherian ring.

(i) R satisfies Serre's condition S_n if for all $P \in \text{Spec}(R)$, $\text{depth}(R_P) \geq \min\{n, \dim(R_P)\}$. Thus, for example, a ring is Cohen-Macaulay if and only if it satisfies S_n for all $n \leq \dim(R)$.

(ii) R satisfies Serre's condition R_n if for all $P \in \text{Spec}(R)$, with $\text{height}(P) \leq n$, R_P is a regular local ring. A ring is regular if and only if it satisfies R_n for all n.

Comments. (i) R is reduced if and only if R satisfies R_0 and S_1. The conditions clearly hold if R is reduced. Suppose the conditions R_0 and S_1 fold. The condition S_1 implies that the associated primes of zero have height zeto, i.e., are the minimal primes of R. The R_0 condition implies that R_q is a field for each minimal prime $q \subseteq R$, and hence $q_q = 0$, for all minimal primes q. Together these conditions give $(0) = q_1 \cap \cdots \cap q_s$, where the q_i are the minimal primes of R. Therefore, R is reduced.

(ii) Even though we have been considering integrally closed domains, the ring R does not have to be an integral domain to be integrally closed. We say that R is integrally closed (as a ring) if R equals the integral closure of R in its total quotient ring. Note however, that if R is integrally closed, then either R is its total quotient ring or R must be reduced - since if $a \in R$ satisfies $a^c = 0$, then for any non-zerodivisor s in R, $\frac{a}{s}$ is an element in the total quotient ring of R, integral over R, yet not in R. With this in mind, one can show that R is integrally closed if and only if R satisfies Serre's conditions R_1 and S_2. The proof of this is almost identical to the proof of Proposition A.

We need a special case of a standard result concerning flatness before proving one of our main results. The general result is known as the *local criterion for flatness* and does not require that S be flat over R - and of course, is more difficult to prove.

Proposition D4. Let $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a flat local homomorphism of Noetherian local rings. A finitely generated S-module M is flat over R if and only if $\mathrm{Tor}_1^R(k, M) = 0$.

Sketch of Proof. If M is flat, then $\mathrm{Tor}_1^R(N, M) = 0$ for all R-modules by applying the long exact sequence in Tor associated to the short exact sequence $0 \to K \to F \to N \to 0$, where F is a free R-module. Conversely, if $\mathrm{Tor}_1^R(N, M) = 0$ for all N, then M is flat since if

$$0 \to A \to B \to C \to 0,$$

is an exact sequence of R-modules, we have and exact sequence

$$\mathrm{Tor}_1^R(C, M) \to A \otimes M \to B \otimes M \to C \otimes M \to 0.$$

Since $\mathrm{Tor}_1^R(C, M) = 0$, the map $A \otimes M \to B \otimes M$ is injective, showing M is flat. We now make a series of reductions.

Step 1. M is flat over R if $\mathrm{Tor}_1^R(N, M) = 0$, for all finitely generated R-modules. This follows since $N = \varinjlim N_i$ is a direct limit of finitely generated R-modules and $\varinjlim \mathrm{Tor}_1^R(N_i, M) = \mathrm{Tor}_1^R(\varinjlim N_i, M)$, so M is flat over R if $\mathrm{Tor}_1^R(N, M) = 0$, for all finitely generated R-modules N.

Step 2. Let N be a finitely generated R-module. Then N has a filtration $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = N$, such that $N_i/N_{i-1} \cong R/P_i$, where each $P_i \subseteq R$ is a prime ideal. If $\mathrm{Tor}_1^R(R/P_i, M) = 0$, for all i, then induction on i together with the long exact Tor sequence applied to the sequences $0 \to N_{i-1} \to N_i \to N_i/N_{i-1} \to 0$, shows that $\mathrm{Tor}_1^R(N_i, M) = 0$ for all i and hence $\mathrm{Tor}_1^R(N, M) = 0$. Thus, it suffices to prove $\mathrm{Tor}_1^R(R/I, M) = 0$, for all ideals $I \subseteq R$.

Step 3. Suppose $\mathrm{Tor}_1^R(R/J, M) = 0$, for all $\mathfrak{m}$-primary ideals $J \subseteq R$. Let $I \subseteq R$ be an ideal. Fix $t \geq 1$. Then $J := I + \mathfrak{m}^t$ is $\mathfrak{m}$-primary. Let $0 \to K \to F \to M \to 0$ be an exact sequence of S-modules with F finitely generated and free over S. Note that this is also an exact sequence of R-modules, and that since S is flat over R, F is flat over R. Let us also note that in this situation $\mathrm{Tor}_1^R(R/J, M) = (JF \cap K)/JK$. To see this, tensor the short exact sequence $0 \to K \to F \to M \to 0$ with R/J. We get the long exact sequence in Tor

$$\cdots \to \mathrm{Tor}_1^R(R/J, F) \to \mathrm{Tor}_1^R(R/J, M) \to K/JK \to F/JF \to M/JM \to 0.$$

$\mathrm{Tor}_1^R(R/J, F) = 0$, since F is flat over R. This shows $\mathrm{Tor}_1^R(R/J, M)$ is isomorphic to the kernel of the map from K/JK to F/IF, which is $(JF \cap K)/JK$. The same argument shows that $\mathrm{Tor}_1^R(R/I, M) = (IF \cap K)/IK$.

Now, $\mathrm{Tor}_1^R(R/J, M) = 0$ implies $JF \cap K = JK$. Thus $IF \cap K \subseteq JF \cap K = (I + \mathfrak{m}^t)K \subseteq (I + \mathfrak{n}^t)K$. Since K is finitely generated and S is local, taking this last intersection over all t shows $IF \cap K \subseteq IK$, and hence $IF \cap K = IK$. Thus, by the comments above we have $\mathrm{Tor}_1^R(R/I, M) = 0$. It now suffices to shows $\mathrm{Tor}_1^R(R/J, M) = 0$ for all $\mathfrak{m}$-primary ideals.

Step 4. Let $J \subseteq R$ be $\mathfrak{m}$-primary. It suffices to prove $\mathrm{Tor}_1^R(N, M) = 0$, for all finite length R-modules N. Proceeding by indeuction on the length, when the length is one, $N \cong k$, and our assumption gives $\mathrm{Tor}_1^R(N, M) = 0$. When N has length greater than one, we can find an R-module $N' \subseteq N$ such that N/N' has length one. We then apply the long exact Tor sequence associated to $0 \to N' \to N \to N/N' \to 0$ to complete the proof. □

Here is an important corollary.

Corollary E4. *Let $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}.l)$ be a flat local homomorphism of local rings. Suppose $\underline{x} = x_1, \ldots, x_r \in S$ have the property that their images in $S/\mathfrak{m}S$ form a regular sequence. Then $\underline{x}$ forms a regular sequence in S and $S/(\underline{x})S$ is flat over R.*

Proof. It suffices to prove the case $r = 1$. So, suppose $x \in S$ is a non-zerodivisor on $S/\mathfrak{m}S$. Take $s \in S$ and suppose $sx = 0$. Then $sx \in \mathfrak{m}S$, so $s \in \mathfrak{m}S$. Let $a_1, \ldots, a_d \in R$ be a minimal generating set for $\mathfrak{m}$. Then we have part of a minimal resolution of $\mathfrak{m}$ over R given by $R^c \xrightarrow{\alpha} R^d \to \mathfrak{m} \to 0$, where the matrix α has entries in $\mathfrak{m}$. Tensoring with S, we preserve exactness and have $S^c \xrightarrow{\alpha \otimes 1} S^d \to \mathfrak{m}S \to 0$, where $\alpha \otimes 1$ is just the matrix α. On the other hand, we may write $s = s_1a_1 + \cdots s_da_d$, with $s_i \in S$. Therefore, $0 = (xs_1)a_1 + \cdots + (xs_d)a_d$.

It follows that the column vector $\begin{pmatrix} xs_1 \\ \vdots \\ xs_d \end{pmatrix}$ belongs to the image of $\alpha \otimes 1$. Thus each $xs_i \in \mathfrak{m}S$. Therefore, by our assumption on x, each $s_i \in \mathfrak{m}S$. Thus, $s \in \mathfrak{m}^2 S$. Therefore we can rewrite s as an S linear combination of of the generators of $\mathfrak{m}^2$. Repeating the argument with the start of a minimal resolution of $\mathfrak{m}^2$ shows $s \in \mathfrak{m}^3 S$, and thus by induction, $s \in \mathfrak{m}^t S$ for all t. Therefore, $s = 0$, since S is local and $\mathfrak{m} \subseteq \mathfrak{n}$. Thus, x is a non-zerodivisor in S.

Now, consider the exact sequence $0 \to S \xrightarrow{x} S \to S/xS \to 0$. Tensoring with k we get:

$$\cdots \to \mathrm{Tor}_1^R(k, S) \to \mathrm{Tor}_1^R(k, S/xS) \to S/\mathfrak{m}S \xrightarrow{x} S/\mathfrak{m}S \to S/(x, \mathfrak{m})S \to 0.$$

In the Tor sequence above, multiplication by x is injective, by the assumption on x, and $\mathrm{Tor}_1^R(k, S) = 0$, since S is flat over R. Therefore $\mathrm{Tor}_1^R(k, S/xS) = 0$, and thus S/xS is flat over R, by Theorem D4, as required. ☐

Here is one of the main results from this section.

Theorem F4. *Let $\phi : R \to S$ be a faithfully flat ring homomorphism.*

(i) If S satisfies S_n, then R satisfies S_n.
(ii) If R satisfies S_n and the fibers $k(p) \otimes S$ satisfy S_n, for all $p \in \mathrm{Spec}(R)$, then S satisfies S_n.
(iii) Statements (i) and (ii) hold for Serre's condition R_n.

Proof. For (i), take $P \in \mathrm{Spec}(R)$ and $Q \in \mathrm{Spec}(S)$ such that Q is minimal over PS. Then S_Q is faithfully flat over R_P and $\mathrm{height}(Q) = \mathrm{height}(P)$. Now suppose $\mathrm{depth}(R_P) = r$. If $x_1, \ldots, x_r$ is a maximal regular sequence in R_P, by faithful flatness, these elements remain a regular sequence in S_Q. Moreover, there exists $c \in R$ with $Pc \in (x_1, \ldots, x_r)R$ and $c \notin (x_1, \ldots, x_r)R$. Therefore, $PSc \in (x_1, \ldots, x_r)S$ and by flatness, $c \notin (x_1, \ldots, x_r)S$. Since $Q^t \subseteq PS$ for some t, Q^t consists of zerodivisors modulo $(x_1, \ldots, x_r)S_Q$. Thus, $\mathrm{depth}(S_Q) = \mathrm{depth}(R_P)$. Since $\dim(R_P) = \dim(S_Q)$, it follows that if $\mathrm{depth}(S_Q) \geq \min\{n, \dim(S_Q)\}$, then $\mathrm{depth}(R_P) \geq \min\{n, \dim(R_P)\}$, and thus R satisfies S_n.

For (ii) suppose R and the fibers $k(p) \otimes S$ satisfy S_n. Let $Q \subseteq S$ be a prime ideal, and set $P := Q \cap R$. We may localize at Q, so that R is a local ring with maximal ideal P and S is local at Q and flatness is preserved. Note that S/PS is a fiber of our original ring homomorphism. Thus, R and S/P satisfy S_n. Let $r := \mathrm{depth}(S/PS)$. Take $\underline{y} = y_1, \ldots, y_r \in S$ such that their images in S/PS form a regular sequence. By Corollary E4, the sequence $\underline{y}$ is a regular sequence in S and $S/(\underline{y})S$ is flat over R. Now take $x_1, \ldots, x_s \in R$, a maximal regular sequence so that $s := \mathrm{depth}(R)$. Since $S/(\underline{y})S$ is flat over R, and $\underline{x} \cdot (S/(\underline{y})S) \neq S/(\underline{y})S$, the sequence $\underline{x}$ is a regular sequence on $S/(\underline{y})S$. Thus, $\underline{y}, \underline{x}$ is a regular sequence in S. Therefore,

$$\mathrm{depth}(S) \geq \mathrm{depth}(R) + \mathrm{depth}(S/PS) \geq \min\{n, \dim(R)\} + \min\{n, \dim(S/PS)\}.$$

Consider the sum on the far right. If n is strictly less than one of $\dim(R)$ or $\dim(S/PS)$, then one of the terms in the sum equals n, so the sum, and hence $\mathrm{depth}(S)$ is greater than $\min\{n, \dim(S)\}$. Suppose both $\dim(R)$ and $\dim(S/PS)$ are less than or equal to n. The sum on the right above becomes $\dim(R) + \dim(S/PS) = \dim(S)$, and thus $\mathrm{depth}(S) \geq \min\{n, \dim(S)\}$, in this case as well. This shows S satisfies S_n.

For part (iii), assume first that S satisfies R_n. Take $P \subseteq R$ a prime ideal with height less than or equal to n. We must show R_P is regular. If we localize S at a prime Q minimal over P, then $\mathrm{height}(Q) = \mathrm{height}(P) \leq n$. If we localize at Q, we may assume that ϕ is a flat, local homomorphism between local rings of the same dimension and that S is a regular local ring. Let k denote the residue field of R and take the start of a minimal free resolution $\cdots \to F_2 \to F_1 \to R \to k \to 0$ of k as an R-module. It suffices to show $F_n = 0$, for some n, for then k will have finite projective dimension over R and thus, R will be a regular local ring. Tensor this resolution with S. Since S is flat over R, the new sequence $\cdots \to F_2 \otimes S \to F_1 \otimes S \to S \to k \otimes S \to 0$ is exact. Moreover, this is a minimal resolution over S. Since S is regular, we must have $F_n \otimes S = 0$, for $n \geq \dim(S)$. In particular, some $F_n \otimes S = 0$, and by faithful flatness, $F_n = 0$, for some n. Thus, R is regular, which is what we want.

Now suppose R and the fibers of ϕ satisfy R_n. As before, we take $Q \in \mathrm{Spec}(S)$ and localize S at Q, so that for $P = Q \cap R$, R is local at P, and hence regular, and the closed fiber S/PS is regular. Now, P is generated by a regular sequence in R, which remains regular in S, by flatness. Moreover, Q/PS is generated by a regular sequence. Putting these sequences together shows that Q is generated by a regular sequence which means S is a regular local ring, which is what we want. □

Here are some immediate corollaries.

Corollary G4. *Let $\phi : R \to S$ be a faithfully flat ring homomorphism. Then:*

(i) S is Cohen-Macaulay if and only if R is Cohen-Macaulay and the fibers $k(p) \otimes S$ are Cohen-Macaulay, for all $p \in \text{Spec}(R)$.

(ii) S is regular if and only if R is regular and the fibers $k(p) \otimes S$ are regular, for all $p \in \text{Spec}(R)$.

Proof. Immediate from Theorem F4. □

Corollary H4. *Let $(R, \mathfrak{m})$ be a local ring. Then:*

(i) $\widehat{R}$ is reduced if and only R is reduced and the formal fibers are reduced. In particular, if R is a local domain, then R is analytically unramified if and only the formal fibers of R satisfy S_1 and R_0.

(ii) $\widehat{R}$ is integrally closed if and only if R is integrally closed and the formal fibers of R are integrally closed. In particular, if R is an integrally closed local domain, then $\widehat{R}$ is an integrally closed domain if and only if the formal fibers of R satisfy S_2 and R_1.

Proof. This follows from Theorem F4 and the characterization of th reduced and integrally closed properties in terms of the Serre conditions S_n and R_n. □

We now state some crucial elements in the definition of an excellent local ring.

Definitions. (i) Let k be a field and A an algebra over k (typically Noetherian). A is said to be *geometrically regular* if for every finite field extension $k \subseteq k'$, $k' \otimes_k A$ is regular.

(ii) The Noetherian ring R is said to be a G-ring if, for every $Q \in \text{Spec}(R)$, the formal fibers of R_Q are geometrically regular.

As we are not going to prove anything of substance with this properties, a number of comments are in order.

Comments. (i) If the k-algebra is geometrically regular, it is clearly regular. On the other hand, it turns out that if A is regular, and k' is a finite separable extension of k, then $k' \otimes_k A$ is automatically regular. Thus if k is a perfect field, any regular k-algebra is geometrically regular. In particular, if k has characteristic zero, then any regular k-algebra is geometrically regular. Therefore, if R contains a field of characteristic zero, then R is a G-ring if and only if the formal fibers of R_Q are regular, for all $Q \in \text{Spec}(R)$.

(ii) Suppose $(R, \mathfrak{m})$ is a local ring. To say that the formal fibers of R are geometrically regular, means that for every prime $p \in \text{Spec}(R)$, the $k(p)$-algebra $k(p) \otimes_R \widehat{R}$ is geometrically regular. For non-local R, to be a G-ring means this property holds for R_Q, for all $Q \in \text{Spec}(R)$.

(iii) Some deep theorems concerning G-rings are:

(a) If $(R, \mathfrak{m})$ is a local ring and the formal fibers of R are geometrically regular, then R is a G-ring. In other words, in the local case, one does not have to check the formal fibers of R_Q, for $Q \in \text{Spec}(R)$.

(b) A complete local ring is a G-ring.

(c) If R is a G-ring, then any finitely generated R-algebra is also a G-ring.

The proofs of these theorems involve a lot of machinery, including modules of differentials and the notions pertaining to *formal smoothness*. Most of the details can be found in Matsumura's first book *Commutative Algebra*. As one might suspect, the difficulties mainly lie in the characteristic $p > 0$ case or the mixed characteristic case.

(iv) A local ring is *excellent* (finally!) if it is a universally catenary G-ring. When R is not local, an additional condition is required for a ring to be excellent, namely that, for any finitely generated R-algebra T, the set of primes $Q \in \text{Spec}(T)$ such that R_Q is regular form an open subset of $\text{Spect}(T)$. It turns out that this condition holds automatically in a local G-ring, though this is also difficult to prove.

(v) An excellent local domain is a Nagata domain. This will follow from the theorem below.

(vi) A finitely generated algebra over an excellent ring is excellent. The difficulty here lies in the transference of the G-ring property. Also, homomorphic images and localizations of excellent rings are excellent.

(vi) It follows from what we have done this semester, and the statements above, that a regular local ring containing a field of characteristic zero is excellent. But we have already seen with Nagata's example that even a DVR in characteristic $p > 0$ need not be excellent. There are examples of regular local rings of mixed characteristic that are not excellent.

(vii) Standard examples of excellent rings include:

(a) Complete local rings (including fields).
(b) Characteristic zero Dedekind domains, including $\mathbb{Z}$.
(c) Finitely generated algebras over rings in (a), (b), and homomorphic images and localizations of these algebras.

(vii) A celebrated theorem of E. Kunz states that if R is a Noetherian ring containing a field of characteristic p, the R is excellent if R is a finite R^p-module. R. Datta and K. Smith recently proved that if R is a domain containing a field of characteristic $p > 0$ and its quotient field K satisfies $[K : K^p] < \infty$, then if R is excellent, then R is finite over R^p.

We close this section with a theorem concerning Nagata rings and formal fibers. We need two preliminary results, the first of which we state as a Remark.

Remark. Let A be a ring , $S \subseteq A$ a multiplicatively closed ate. For A_S-modules M and N, we have $M \otimes_A N = M \otimes_{A_S} N$ and for A-modules M, N, we have $M_S \otimes_A N = (M \otimes_A N)_S$. We will use these properties below without comment.

The following lemma extends what we already know in the domain case to the case of reduced local rings.

Lemma I4. *Let $(R, \mathfrak{m})$ be a reduced Nagata ring with total quotient ring K. Let T be a finite, integral extension of K such that T is also reduced. Let S denote the integral closure of R in T. Then S is a finite R-module.*

Proof. Note that both K and T are direct sums of fields. Let $P \subseteq T$ be a minimal (and also, maximal) prime ideal. Then $K/(P \cap K) \subseteq T/P$ is a finite extension of fields. Since $R/(P \cap R)$ is a Nagata ring, the integral closure of $R/(P \cap R)$ in T/P is a finite $R/(P \cap R)$-module. Since $S/(S \cap P)$ is contained in that integral closure, $S/(P \cap S)$ is a finite module over $R/(P \cap R)$, and hence, also a finite R-module. If we let $P_1, \dots P_r$ denote the minimal primes of T then $T = T/P_1 \oplus \cdots \oplus T/P_r$ (by the Chinese remainder theorem), and we have $S \hookrightarrow S/(P_1 \cap S) \oplus \cdots \oplus S/(P_r \cap S) \subseteq T$. Since each $S/(P_i \cap S)$ is a finite R-module, S is a finite R-module, which is what we want. □

Definition. Let k be a field. A k-algebra A is said to be *geometrically reduced* if $k' \otimes_k A$ is reduced for all finite extensions k' of k.

Theorem J4. *Let $(R, \mathfrak{m})$ be a local domain with quotient field K. Then R is a Nagata ring if and only if its formal fibers are geometrically reduced.* In particular, if R is an excellent local domain, then R is a Nagata ring.

Proof. Assume that R is a Nagata ring and let $p \in \operatorname{Spec}(R)$. Set $U := R \backslash p$. We need to show that $k(p) \otimes)_R \widehat{R} = (\widehat{R}/p\widehat{R})_U$ is geometrically reduced. Note that this is the generic formal fiber of R/p which is a Nagata ring. Thus, we may replace R/p by R and begin again assuming that R is a Nagata ring and show that its generic formal fiber $K \otimes \widehat{R} = \widehat{R}_U$ is geometrically reduced, where U is the set of non-zero elements of R.

Let K' be a finite field extension of K. We want $K' \otimes_K \widehat{R}_U$ to be reduced. Let $\tilde{K}$ denote the total quotient ring of $\widehat{R}$. We first note that $K \otimes_K \widehat{R}_U \hookrightarrow K' \otimes_K \tilde{K}$, since $K \subseteq K'$ is a faithfully flat extension of K-modules. Thus, it suffices to show $K' \otimes_K \tilde{K}$ is reduced. But

$$K' \otimes_K \tilde{K} = K' \otimes_R \tilde{K} \subseteq \mathrm{QR}(K' \otimes_R \widehat{R}),$$

where 'QR' denotes quotient ring. Thus, it suffices to show that $\mathrm{QR}(K' \otimes_R \widehat{R})$, and hence $K' \otimes_R \widehat{R}$ is reduced. Now, let S denote the integral closure of R in K'. Then S is finite over R, so $\widehat{S} = S \otimes_R \widehat{R}$ and S is also a

Nagata ring by Comment 2 after the definition of Nagata ring on page 1 above. Thus, $\widehat{S}$ is reduced, since a Nagata local domain is analytically unramified (by Theorem H). Therefore $\mathrm{QR}(\widehat{S})$ is reduced. On the other hand, we have

$$K' \otimes_R \widehat{R} = \mathrm{QR}(S) \otimes_R \widehat{R} \subseteq \mathrm{QR}(S \otimes_R \widehat{R}) = \mathrm{QR}(\widehat{S}),$$

which shows $K' \otimes_R \widehat{R}$ is reduced, which is what we want.

Conversely, suppose the formal fibers of R are geometrically reduced. Let $P \in \mathrm{Spec}(R)$. We must show R/P satisfies N_2. Since the generic formal fiber of R/P is geometrically reduced, as before, we may replace R/P by R, assume that R is a local domain with quotient field K and that the generic formal fiber $K \otimes_R \widehat{R} = \widehat{R}_U$ is geometrically reduced, where U is the set of non-zero elements in R. Let K' be a finite field extension of K and S be the integral closure of R in K'. We must show S is a finite R-module. For this, it suffices to show that $S \otimes_R \widehat{R}$ is a finite $\widehat{R}$-module. Note: we do not yet know $S \otimes_R \widehat{R} = \widehat{S}$. We have

$$S \otimes_R \widehat{R} \subseteq K' \otimes_R \widehat{R} \subseteq K' \otimes_R \widehat{R}_U = K' \otimes_K \widehat{R}_U,$$

with the last ring on the right being reduced by assumption. Thus, $K' \otimes_R \widehat{R}$ is reduced and hence its's quotient ring T is reduced.

On the other hand,

$$\widehat{R} \hookrightarrow \widehat{R}_U = K \otimes_R \widehat{R} \hookrightarrow K' \otimes_R \widehat{R}$$

which shows that $\widehat{R}$ and its quotient ring $\tilde{K}$ are reduced. Since T is a finite extension of $\tilde{K}$, we are in the situation of Lemma I4 above. $\widehat{R}$ is a reduced Nagata ring (since a complete local ring is a Nagata ring), so its integral closure in T is a finite $\widehat{R}$-module. Since $S \otimes_R \widehat{R}$ is contained in this ring, $S \otimes_R \widehat{R}$ is a finite $\widehat{R}$-module, which completes the proof. □

5. The Rees multiplicity theorem

In this section, we change direction entirely to focus on multiplicities in local rings, with the goal of proving the celebrated theorem of Rees, which state the following: Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring and $J \subseteq I$ two $\mathfrak{m}$-primary ideals satisfying $e(J) = e(I)$. Then $\overline{J} = \overline{I}$. Here we are writing $e(I)$ for the multiplicity if I.

Recall that if $(R, \mathfrak{m})$ is a local ring with $\dim(R) = d$, and $I \subseteq R$ is an $\mathfrak{m}$-primary ideal, one way to define $e(I)$ is as follows:

$$e(I) = \lim_{n \to \infty} \frac{d!}{n^d} \cdot \lambda(R/I^n),$$

where we use $\lambda(-)$ to denote the length of a finite length R-module.

We will develop the definition and basic properties of multiplicities below. In order to prove the theorem of Rees, will will start with some preliminaries on integral closure and then present the standard background material on the multiplicity of an $\mathfrak{m}$-primary ideal in a local ring.

We begin with:

Definition and Comments. An integral domain V with quotient field K is a *valuation domain* if for every $x \in K$, either $x \in V$ or $x^{-1} \in V$. Note that a DVR W is easily seen to be a valuation domain, since every element of W has the form $u\pi^n$, for $u \in W$ a unit and $\pi \in W$ the uniformizing parameter of W. The following hold for a valuation domain V:

(i) Every finitely generated ideal of V is principal. To see this, it suffices to show any two-generated ideal is principal, and to see this it suffices to see that if $a, b \in V$ are nonzero, then either $a \in bV$ or $b \in aV$. But, by definition, either $\frac{a}{b} \in V$ or $\frac{b}{a} \in V$, which gives what we want. NOTE: A valuation domain does not have be a Noetherian. In fact, any Noetherian valuation domain is a DVR.

(ii) A valuation domain has a unique maximal ideal. To see this, let $\mathfrak{m}_V$ denote the set of non-units of V. Clearly $va \in \mathfrak{m}_V$ for all $v \in V$ and $a \in \mathfrak{m}_V$. If $a, b \in \mathfrak{m}_V$ then $a \in bV$ or $b \in aV$, by the previous item. Say, $a \in bV$, so $a = bv$, some $v \in V$. Then $a + b = (v + 1)b \in \mathfrak{m}_V$, so $\mathfrak{m}_V$ is closed under addition. Thus, $\mathfrak{m}_V$ is an ideal, and is therefore the unique maximal ideal of V.

(iii) V is integrally closed. To see this, Suppose $x \in K$ is integral over V. We have an equation of the form:

$$x^n + v_1x^{n-1} + \cdots + v_n = 0,$$

with each $v_j \in V$. Since V is a valuation domain, either x or x^{-1} belongs to V. Suppose $x^{-1} \in V$. Multiply the equation above by x^{-n} to get

$$1 + v_1x^{-1} + \cdots + v_nx^{-n} = 0.$$

Solving for 1 in this equation, we can write $1 = vx^{-1}$, for some $v \in V$. This shows x^{-1} is a unit in V, and hence its inverse x is in V. Thus, V is integrally closed.

(iv) Every ideal in a valuation domain is integrally closed. To see this, let $J \subseteq V$ be an ideal, and take $b \in \overline{J}$. Then b in integral over a finitely generated ideal $J_0 \subseteq J$. By (i) J_0 is principal, and by (iii) V is integrally closed. Thus, $\overline{J_0} = J_0$, and hence $b \in J_0 \subseteq J$.

Our first goal is to characterize the integral closure of powers of an ideal in a Noetherian ring in terms of discrete valuation rings.

Proposition A5. *Let R be a Noetherian domain with quotient field K and $J \subseteq R$ an ideal. Then*

$$\overline{J} = \bigcap_V (JV \cap R) = (\bigcap_V JV) \cap R,$$

where the intersection runs through the DVRs between R and K.

Proof. Let $b \in \overline{J}$. From (iv) above $b \in JV$, for all V. Conversely, suppose $b \notin \overline{J}$. We must find a DVR between R and K with $b \notin JV$. Suppose $J = (a_1, \ldots, a_d)R$, and set $S := R[\frac{a_1}{b}, \ldots, \frac{a_d}{b}]$. Set $L := (\frac{a_1}{b}, \ldots, \frac{a_d}{b})S$. We claim $L \neq S$. Suppose $L = S$. Then there exists a polynomial $f(x_1, \ldots, x_d)$ with coefficients in R such that $f(\frac{a_1}{b}, \ldots, \frac{a_d}{b}) = 1$. Note that if $x_1^{e_1} \cdots x_d^{e_d}$ is a monomial in $f(x_1, \ldots, x_d)$ of degree n, then $b^N \cdot (\frac{a_1}{b})^{e_1} \cdots (\frac{a_d}{b})^{e_d} \in b^{N-n}J^n$, for all $N \geq n$. Thus, if N is the largest degree of a monomial in $f(x_1, \ldots, x_d)$ and we multiply $f(\frac{a_1}{b}, \ldots, \frac{a_d}{b}) = 1$ by b^N and bring b^N to the left hand side of the resulting equation, we have an equation of integral dependence of b on J, contrary to our choice of b. Thus, L is a proper ideal of S.

Now, take a prime ideal $P \subseteq S$ containing L. Then by Corollary G2, there exists a DVR V between S and its quotient field, which is K, such that $\mathfrak{m}_V \cap S = P$. Thus, the elements $\frac{a_i}{b}$ are non-units in V. If b were in JV, then for a_i with $JV = a_iV$, we would have $b \in a_iV$. But then $\frac{b}{a_i} \in V$, would be a contradiction. Therefore $b \notin J$, and the proof is complete. □

Remark. Let A be an integral domain, not necessarily Noetherian, with quotient field K. Then for any prime ideal $P \subseteq A$, using Zorn's lemma, one can show that there exists a valuation domain V (more than likely not a DVR) such that $\mathfrak{m}_V \cap A = P$. The proof above, together with the comments above, show that $\overline{J} = \bigcap_V (JV \cap A)$, where the intersections runs through all valuation domains between A and K.

Corollary B5. *Let R be a Noetherian domain and $I = (a_1, \ldots, a_d)R$ and ideal. Set $T_i = R[\frac{a_1}{a_i}, \ldots, \frac{a_d}{a_i}]$. Then, for all $n \geq 1$, $\overline{I^n} = \bigcap_{1 \leq i \leq d} (\overline{I^nT_i} \cap R)$.*

Proof. Clearly $\overline{I^n} \subseteq \bigcap_{1 \leq i \leq d} (\overline{I^nT_i} \cap R)$. Conversely, suppose $b \in \bigcap_{1 \leq i \leq d} (\overline{I^nT_i}. \cap R)$. Let V be a DVR between R and its quotient field. If $IV = a_iV$, then $a_j \in a_iV$, for all $j \neq i$. Thus each fraction $\frac{a_j}{a_i} \in V$. Therefore $T_i \subseteq V$. But now, $b \in \overline{I^nT_i} \subseteq \overline{I^nV} = I^nV$. Since this holds for all DVRs between R and its quotient field, $b \in \overline{I^n}$, by Proposition A4. □

Theorem C5. *Let R be a Noetherian domain with quotient field K and $I \subseteq R$ be an ideal. Then there exist finitely many DVRs $V_1, \ldots, V_r$ between R and K such that for any $n \geq 1$, $\overline{I^n} = \bigcap_{i=1}^r (I^nV_i \cap R)$.*

Proof. Let $I := (a_1, \ldots, a_d)R$ and set $T_i := R[\frac{a_1}{a_i}, \cdots, , \frac{a_d}{a_i}]$, for all $1 \leq i \leq d$. Take $n \geq 1$, fix $1 \leq i \leq d$ and let T_i' denote the integral closure of T_i. Then T_i' is a Krull domain and thus, from our work in Section 2 we have:

(i) There exist finitely many height one primes $Q_1, \ldots, Q_s \subseteq T_i'$ containing a_i^n, which are exactly the height one primes containing a_i.
(ii) $a_i^nT_i' = (a_i^nW_1 \cap T_i') \cap \cdots \cap (a_iW_s \cap T_i')$, where each $W_j = (T_i')_{Q_j}$.
(iii) Each W_j is a DVR.

Since $\overline{a_i^n T_i} = a_i^n T_i' \cap T_i$, we have

$$\overline{I^n T_i} = \overline{a_i^n T_i} = a_i^n T_i' \cap T_i = (a_i^n W_1 \cap \cdots \cap a_i^n W_s) \cap T_i.$$

Therefore,

$$\overline{I^n T_i} \cap R = (a_i^n W_1 \cap \cdots \cap a_i^n W_s) \cap R = (I^n W_1 \cap R) \cap \cdots \cap (I^n W_s \cap R),$$

since $I^n W_j = a_i^n W_j$, for all j. If we do this for each i, and collect all of the resulting DVRs associated to each $a_i T_i'$, and call them $V_1, \ldots, V_r$, then the conclusion of the theorem follows from Corollary B5.

Remark. The DVRs $V_1, \ldots V_r$ constructed in the proof of Theorem C5 are called the *Rees valuation rings of* I, and are uniquely determined as the smallest collects of DVRs between R and K for which the conclusion of Theorem C5 holds.

We next want to improve the conclusion of Theorem C5 in the case that R is a local domain satisfying the dimension formula and the generators of I form a system of parameters. The next proposition is a special case of a result to E.D. Davis.

Proposition D5. *Let $(R, \mathfrak{m}, k)$ be a local domain and $a_1 \ldots, a_d$ a system of parameters. Fix $1 \leq i \leq d$ and set $T_i := R[\frac{a_1}{a_i}, \cdots, , \frac{a_d}{a_i}]$. Then $\mathfrak{m}T_i$ is a height one prime and the residue classes of $\frac{a_1}{a_i}, \ldots, \widehat{i}, \ldots, \frac{a_d}{a_i}$ are algebraically independent over k, i.e., $T_i/\mathfrak{m}T_i$ is isomorphic to a polynomial ring in $d-1$ variables over k.*

Proof. It suffices to prove the case $i = 1$. Set $T := T_1$ and $S := R[x_2, \ldots, x_d]$, the polynomial ring in $d-1$ variables over R and let P denote the kernel of the natural ring homomorphism from S to T that takes each x_i to $\frac{a_i}{a_1}$, so $P \cap R = 0$. Let L denote the ideal of S generated by $a_1x_2 - a_2, \ldots, a_1x_x - a_d$, so that $L \subseteq P$. Let us note the following: If we invert a_1, then S_{a_1} is the polynomial ring in $d-1$ variables over R_{a_1} and $T_{a_1} = R_{a_1}$. The induced ring homomorphism from $S_{a_1} \to T_{a_1}$ is now just obtained by evaluating any $g(x_2, \ldots, x_d) \in S_{a_1}$ at $\frac{a_2}{a_1}, \ldots, \frac{a_d}{a_1} \in R_{a_1}$. The kernel of an evaluation map is alway just the expected kernel, in this case, $L_0 = (x_2 - \frac{a_2}{a_1}, \ldots, x_d - \frac{a_d}{a_1})S_{a_1}$. Now, clearly $LS_{a_1} = L_0S_{a_1}$, while on the other hand, the kernel of the induced map is P_{a_1}. Thus $P_{a_1} = L_{a_1}$, and hence $P = L_{a_1} \cap S$. In other words, $f(x_2, \ldots, x_d) \in P$ if and only if $a_1^c \cdot f(x_2, \ldots, x_d) \in L$, for some $c \geq 1$.

So: take $f(x_2, \ldots, x_d) \in P$ and let $Q \subseteq S$ be a prime minimal over L. Then $a_1^c \cdot f(x_2, \ldots, x_d) \in L \subseteq Q$. Suppose $a_1^c \in Q$. Then $a_1 \in Q$ and hence $a_1, \ldots, a_d \in Q$. But this is a contradiction, since on the one hand height$(Q) \leq d-1$, while on the other hand $a_1, \ldots, a_d$ generate an ideal of height d in R, and hence also in S. Thus, $a_1 \notin Q$, so $f(x_2, \ldots, x_d) \in Q$. Thus, $P \subseteq Q$, which shows that P is the unique minimal prime of L. Now, since $\mathfrak{m}S$ contains L, we have $P \subseteq \mathfrak{m}S$ Thus $\mathfrak{m}T = \mathfrak{m}S/P$ is a prime ideal. In addition, for some $n \geq 1$,

$$\mathfrak{m}^n \subseteq (a_1, \ldots, a_d)S = (a_1, L)S \subseteq (a_1, P)S,$$

which shows $\mathfrak{m}^n \subseteq a_1T$. Thus height$(\mathfrak{m}T) = 1$, and in fact, $\mathfrak{m}T$ is the unique height one prime in T containing a_1. Finally,

$$T/\mathfrak{m}T = (S/P)/(\mathfrak{m}S/P) \cong S/\mathfrak{m}S \cong k[x_2, \ldots, x_d],$$

the polynomial ring in $d-1$ variables over k. □

The following proposition due to DK plays a key role in a theorem below concerning multiplicities.

Proposition E5. *Let $(R, \mathfrak{m}, k)$ be a local domain and $I = (a_1, \ldots, a_d)R$ an ideal generated by a system of parameters. Assume R satisfies the dimension formula. Set $S := R[\frac{a_2}{a_1}, \cdots, \frac{a_d}{a_1}]_{\mathfrak{m}R[\frac{a_2}{a_1}, \cdots, \frac{a_d}{a_1}]}$. Let $Q_1, \ldots, Q_s$ be the height one primes in S'. Then for all $n \geq 1$, $\overline{I^n} = (I^nV_1 \cap R) \cap \cdots \cap (I^nV_s \cap R)$, where $V_i := (S')_{Q_i}$, for each i.*

Proof. By Theorem C5, we just have to show that $V_1, \ldots, V_s$ is the complete set of Rees valuation rings of I. For each $1 \leq i \leq d$, set $T_i := R[\frac{a_1}{a_i}, \cdots, , \frac{a_d}{a_i}]$, so that $S = (T_1)_{\mathfrak{m}T_1}$. By Proposition D5, $\mathfrak{m}T_i$ is a height one prime. Let $U_i \subseteq T_i$ be the multiplicatively closed subset generated by $\frac{a_1}{a_i}, \ldots, \frac{a_d}{a_i}$. Then

$$(T_i)_{U_i} = T_i[(\frac{a_1}{a_i})^{-1}, \ldots, (\frac{a_d}{a_i})^{-1}] = R[\frac{a_1}{a_i}, \ldots, \frac{a_d}{a_i}, \frac{a_i}{a_1}, \ldots, \frac{a_i}{a_d}].$$

Let $2 \leq j \leq d$ and $i \neq 1$. If $j = i$, then $\frac{a_1}{a_j}, \frac{a_j}{a_1} \in (T_i)_{U_i}$. If $j \neq i$, $\frac{a_j}{a_1} = \frac{a_j}{a_i} \cdot \frac{a_i}{a_1} \in (T_i)_{U_i}$ and $\frac{a_1}{a_j} = \frac{a_1}{a_i} \cdot \frac{a_i}{a_j} \in (T_i)_{U_i}$, which shows that $(T_1)_{U_1} \subseteq (T_i)_{U_i}$. The same argument shows $(T_i)_{U_i} \subseteq (T_1)_{U_1}$, and thus $(T_i)_{U_i} = (T_1)_{U_1}$, for all i. By Proposition D5, $U_i \cap \mathfrak{m}T_i = \emptyset$, for all i, since the images of the elements $\frac{a_i}{a_i}$ in $T_i/\mathfrak{m}T_i$ are algebraically independent over k. Thus $(T_i)_{\mathfrak{m}T_i} = ((T_i)_{U_i})_{\mathfrak{m}(T_i)_{U_i}}$ for all i, from which we infer $(T_i)_{\mathfrak{m}T_i} = S$, for all i.

Now let V be a Rees valuation ring of I. Then for some $1 \leq i \leq d$, V is obtained by localizing T_i' at a height one prime Q containing a_i, so that $\mathfrak{m}_V = Q_Q$. Thus, by Proposition D3, $Q \cap T_i \in \overline{A^*}(a_iT_i)$. Since R satisfies the dimension formula, T_i also satisfies the dimension formula, by Observation 1 following Corollary Q3. Thus, by Proposition L3, height$(Q) = 1$. Since $\mathfrak{m}T_i$ is the only height one prime in T_i containing a_i, $\mathfrak{m}T_i = Q \cap T_i$. Thus $S = (T_i)_{\mathfrak{m}T_i} \subseteq (T_i')_Q = V$. Since V is integrally closed $S' \subseteq V$. Since $\mathfrak{m}_V \cap S = \mathfrak{m}S$, $\mathfrak{m}_V \cap S'$ must contract to $\mathfrak{m}S$, therefore $\mathfrak{m}_V \cap S' = Q_j$, for some $1 \leq j \leq s$. It follows that $V_j \subseteq V$. However, there are no rings strictly between a DVR and its quotient field, so we must have $V_j = V$, which is what we want. □

Our next Theorem is a special case of one of the main theorems in the theory of reductions of ideals due to Northcott and Rees. Their famous 1953 paper titled *Reductions of ideals in local rings* is one of the most frequently cited papers in commutative algebra. For the theorem below, we will use the Noether Normalization Theorem, one version of which is the following: Let k be an infinite field and B a finitely generated, graded k algebra, which is generated over k by homogenous elements of degree one. If $\dim(B) = r$, then there exist $b_1, \ldots, b_r \in B_1$, such that $b_1, \ldots, b_r$ are algebraically independent over k and B is a finite module over $A := k[b_1, \ldots, b_r]$.

Theorem F5. *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field k. Then, for any $\mathfrak{m}$-primary ideal $I \subseteq R$, there exists an ideal J, generated by a system of parameters, such that $\overline{J} = \overline{I}$.*

Proof. Let $\mathcal{R}$ denote the Rees ring of R with respect to I, so that $R := R[It] = R \oplus It \oplus I^2t^2 \oplus \cdots$. The k-algebra $B := \mathcal{R}/\mathfrak{m}\mathcal{R}$ is a finitely generated, graded k-algebra generated by homogeneous elements of degree one over k. Note, that as a graded k-algebra, $B = k \oplus I/\mathfrak{m}I \oplus I^2/\mathfrak{m}I^2 \oplus \cdots$. By Noether's Normalization Lemma, there exist $b_1, \ldots, b_r \in B_1$, such that $b_1, \ldots, b_r$ are algebraically independent over k and B is a finite module over $A := k[b_1, \ldots, b_r]$. Note that each $b_i = \overline{a_i}$, for some $a_i \in I \backslash \mathfrak{m}I$.

We are now in an Artin-Rees like situation. B is a finite, graded module over the graded ring A, and as such we can take finitely many homogenous elements $c_1, \ldots, c_s \in B$ that generate B as an A-module. If n is the maximum degree of any c_j, then it follows that for all $t \geq 0$, $B_{n+t} = A_t \cdot B_n$. In particular, $B_{n+1} = A_1B_n$. Interpreting this in terms of R, we have $I^{n+1} \subseteq JI^n + \mathfrak{m}I^{n+1}$, where $J = (a_1, \ldots, a_r)R$. We note two things: (i) $I^{n+1} \subseteq JI^n$, by Nakayama's lemma and (ii) $\dim(B) = \dim(R)$. The second of these follows since (say by Atiyah-MacDonald, Chapter 11), $\mathcal{R}/I\mathcal{R} = \bigoplus_{\mathfrak{n} \geq 0} I^n/I^{n+1}$, the associated graded ring of R with respect to I, has dimension equal to $\dim(R)$. Since $\mathfrak{m}^n\mathcal{R} \subseteq I\mathcal{R} \subseteq \mathfrak{m}\mathcal{R}$, it follows that $\mathcal{R}/I\mathcal{R}$ and B have the same dimension. Thus, $r = d$ and J is generated by a system of parameters. Since $J \subseteq I$ and $JI^n \subseteq I^{n+1}$, we have $JI^n = I^{n+1}$.

We now show $\overline{J} = \overline{I}$. Now let $q \subseteq R$ be a minimal prime ideal. By Lemmma B3, it is enough to show that the image of I in R/q and the image of J in R/q have the same integral closure. Since the identity $I^{n+1} = JI^n$ also holds modulo q, we may replace R/q by R and assume that R is an integral domain. Let V be a DVR between R and its quotient field. Then, $I^{n+1}V = JI^nV$. Since the ideals IV and JV are principal ideals, we may cancel I^n from both sides of this equation to get $IV = JV$. Since this holds for all DVRs V, we have $\overline{I} = \overline{J}$, by Theorem A5. □

We now begin our discussion of multiplicities. We will prove a standard result about the existence of Hilbert polynomials associated to graded modules over a graded ring. The following is a key technical lemma needed for the induction part of the proof of the existence of Hilbert polynomials. For this lemma, we need the following remark concerning primary decomposition in modules. We will use the fact below that the zero submodule of M is in intersection of primary submodules, each of which is a graded submodule of M.

Remark. Let $A/$ be a Noetherian ring and M a finitely generated A-module. A submodule $N \subseteq M$ is said to be *P-primary*, for the prime ideal $P \subseteq A$ if $\mathrm{Ass}_A(M/N) = P$. Note that if $I \subseteq A$ is an ideal, then this is saying the same thing as I is P-primary, since then $\mathrm{Ass}_A(R/I)$ is P-primary. Moreover, since a prime minimal over the annihilator of a finitely generated A-module is an associated primes, if N is P-primary, then $P^S \cdot (M/N) = 0$, for some S. Further, if M is a finitely generated A-module, then the zero submodule of M has a primary decomposition in terms of primary submodules of M. Finally, if A is a graded ring and M, N are graded modules, then the associated primes of M/N are homogeneous.

Lemma G5. *Let $A = \bigoplus_{n\geq 0} A_n$ be a finitely generated R-algebra, where $A_0 = (R, \mathfrak{m}, k)$ is a local ring with infinite residue field. Write $\mathcal{M}$ for the homogeneous maximal ideal $(\mathfrak{m}, A_+)A$. We assume that A is a standard, graded R-algebra, i.e., $A = R[A_1]$. Let $M = \bigoplus_{n\geq 0} M_n$ be a finitely generated, graded A-module. If $Ass(M) \neq \mathcal{M}$, then there exists $f \in A_1$ and $c > 0$ such that $(0 :_M f)_n = 0$, for all $n \geq c$. In other words, elements in M annihilated by f are concentrated in degrees less than c.*

Proof. Let $(0) = N_1 \cap \cdots \cap N_r \cap N_{r+1} \cap \cdots \cap N_2$ be a primary decomposition, where for $1 \leq i \leq r$, $\text{Ass}_A(M/N_i) = Q_i$ does not contain A_+ and for $r+1 \leq i \leq s$, $A_+ \subseteq Q_i = \text{Ass}(M/N_i)$. We claim there exists $f \in A_1$ such that $f \notin Q_1 \cup \cdots \cup Q_r$. Suppose the claim holds. Take $c > 0$ such that for $r+1 \leq i \leq s$, $(M/N_i)_n = 0$, for $n \geq c$. This is possible since each M/N_i is annihilated by a power of A_+.[2] Then for $n \geq c$, if $b \in M_n$ and $fb = 0$, then, on the one hand, $fb \in N_i$ for all $1 \leq i \leq r$, which, by the choice of f, implies $b \in N_i$, all i. On the other hand, by the choice of c, $b \in N_{r+1} \cap \cdots \cap N_s$. Thus b belongs to all of the primary components of (0), so $b = 0$.

For the claim, consider the k-vector space $V := A_1/\mathfrak{m}A_1$ and the subspaces $L_i := ((Q_i)_1 + \mathfrak{m}A_1)/\mathfrak{m}A_1$, $1 \leq i \leq r$. These are proper subspaces of V, for if say, $L_i = V$, then $A_1 = (Q_i)_1 + \mathfrak{m}A_1$. Since A is a standard graded algebra, this implies $A_+ \subseteq Q_i + \mathcal{M}A_+$, which by Nakayama's lemma (the graded version) implies $A_+ \subseteq Q_i$, a contradiction. Thus, the subspaces L_i are proper subspaces of V, and since k is infinite, there exists $\overline{f} \in V \backslash (L_1 \cup \cdots \cup L_r)$. Thus, $f \in A_1$, but f is not in any Q_i, as required. $\square$

Definition and comments. One can draw a similar conclusion to Lemma 5G if k is not infinite. One uses the homogeneous form of prime avoidance. This, together with the definition of the Q_j, impliy that $A_+ \nsubseteq Q_1 \cup \cdots \cup Q_r$, and thus, there exists a homogeneous ring element f not in $Q_1 \cup \cdots \cup Q_r$, and the conclusion of the lemma still holds for this f. However, f may not be homogeneous of degree one. Such elements are called *superficial elements*, and if $f \in A_d$, then f is a *superficial element of degree d.* Thus, superficial elements of some positive degree exist, but superficial elements of degree one need not always exist.

Facts about numerical polynomials. A *numerical polynomial* is a polynomial $P(x) \in \mathbb{Q}$ such that $P(n) \in \mathbb{Z}$, for all $n \in \mathbb{Z}$ (or equivalently, all $n \in \mathbb{N}$). Note that the polynomial associated to the binomial coefficient, $\binom{x+d}{x} := \frac{1}{d!} \cdot (x+d)(x+d-1) \cdots (x+1)$ is a numerical polynomial of degree d. A function $f : \mathbb{Z} \to \mathbb{Z}$ is said to *agree with a numerical polynomial for $n >> 0$* if there exists $n_0 \in \mathbb{N}$ and a numerical polynomial $F(x)$ such that $f(n) = F(n)$, for all $n \geq n_0$. We will use the following two facts.

(i) Any numerical polynomial $P(x)$ of degree d can be written uniquely as

$$P(x) = e_0 \binom{x+d}{d} + e_1 \binom{x+d-1}{d-1} + \cdots + e_d \binom{x+0}{0},$$

with the $e_j \in \mathbb{Z}$. To see this, first note that since each $\binom{x+d}{d}$ has degree d, these polynomials form a basis for $\mathbb{Q}[x]$ as a vector space over $\mathbb{Q}$. Thus any polynomial in $\mathbb{Q}[x]$ can be written uniquely as a $\mathbb{Q}$-linear combination of the $\binom{x+d}{x}$. However, if $P(x)$ is a numerical polynomial, then one can show by induction on the degree of $P(x)$ that the coefficients e_j above must be integers. Note also, that if $P(n) \in \mathbb{N}$, for $n \in \mathbb{N}$, then $e_0 \in \mathbb{N}$.

(ii) Suppose $f : \mathbb{N} \to \mathbb{N}$ has the property that $f(n+1) = f(n)$ agrees with a numerical polynomial of degree d for $n >> 0$. The $f(n)$ agrees with a numerical polynomial of degree $d+1$ for $n >> 0$. To see this, suppose suppose $f(n+1) - f(n) = P(n)$, for $n >> 0$, where

$$P(x) = \sum_{j=0}^{d} e_j \binom{x+d-j}{d-j}.$$

[2]To see this, suppose $A_+^e \cdot (M/N) = 0$. Then $X := M/N$ is a finitely generated graded module over $B := A/A_+^e$. Since A is a standard graded algebra, $B_n = 0$, for all $n \geq e$. Since there exists n_0 such that $X_n = B_{n-n_0} X_{n_0}$, for all $n \geq n_0$, $X_n = 0$, for $n - n_0 \geq e$.

Set

$$F(x) := \sum_{j=0}^{d} e_j \binom{x+d-j}{d-j+1}.$$

Then, for $n >> 0$,

$$\begin{aligned} F(n+1) - F(n) &= \sum_{j=0}^{d} e_j \{ \binom{n+1+d-j}{d-j+1} - \binom{n+d-j}{d-j+1} \} \\ &= \sum_{j=0}^{d} e_j \binom{n+d-j}{d-j} \\ &= f(n+1) - f(n). \end{aligned}$$

It follows that $(F-f)(n+1) - (F-f)(n) = 0$, for $n >> 0$. Thus, $(F-f)(n) = c$, a constant for $n >> 0$. Therefore, $f(n) = F(n) - c$, for $n >> 0$, which shows that $f(n)$ agrees with a numerical polynomial of degree $d+1$ for $n >> 0$.

We need one more observation before proving the main theorem concerning the existence of Hilbert polynomials.

Comments on extending the residue field. (i) Let $(R, \mathfrak{m}, k)$ be local ring with finite residue field k. Take an indeterminate y and consider the ring the ring $R[y]_{\mathfrak{m}R[y]}$. This ring is denoted $R(y)$. Then $R(y)$ is a faithfully flat local extension of R whose maximal ideal is $\mathfrak{m}R(y)$ and whose residue field $k(y)$ is infinite. Let $U \subseteq V$ be two R-modules such that $\lambda(V/U) = 1$. Then there is an exact sequence

$$0 \to U \to V \to k \to 0.$$

If we tensor this exact sequence with $R(y)$, we have

$$0 \to U \otimes R(y) \to V \otimes R(y) \to k(y) \to 0,$$

where $k(y)$ is the residue field of $R(y)$. Thus, $\lambda_{R(y)}(V \otimes R(y)/U \otimes R(y)) = 1$. It follows that if C is a finite length R-module having length c, then $C \otimes R(y)$ is a finite length $R(y)$-module with length c. In particular, if $J \subseteq R$ is an $\mathfrak{m}$-primary ideal, then since $JR(y) = J \otimes R(y)$, $\lambda(R/J) = \lambda(R(y)/JR(y))$.

(ii) Now suppose A is a standard graded ring, finitely generated as an algebra over $A_0 = (R, \mathfrak{m}, k)$. Then $\tilde{A} := A \otimes_R R(y)$ is a standard graded ring, finitely generate as an algebra over $R(y)$ and if M is a finite, graded A-module, then $\tilde{M} := M \otimes_R R(y)$ is a finite, graded $\tilde{A}$-module. It is straightforward to show that since $R(y)$ is faithfully flat over R, then $\tilde{A}$ is faithfully flat over A. In fact, if $U = R[y] \backslash \mathfrak{m}R[y]$, then $\tilde{A}$ can be identified with $A[y]_U$. Now suppose $\dim(M) = d$. Then $\dim(A/J) = d$, where J is the annihilator of M. If we take a set of generators $x_1, \ldots, x_r$ of M, then we have an exact sequence

$$0 \to J \to A \xrightarrow{\phi} M \oplus \cdots \oplus M,$$

where $\phi(a) = (ax_1, \ldots, ax_d)$, for all $a \in A$. If we tensor this exact sequence with $\tilde{A}$, we have an exact sequence

$$0 \to J \otimes_A \tilde{A} \to \tilde{A} \xrightarrow{\phi \otimes 1} \tilde{M} \oplus \cdots \oplus \tilde{M},$$

where $\phi \otimes 1$ takes $\tilde{a} \in \tilde{A}$ to $\tilde{a}(x_i \otimes 1)$ in each component. Since the $x_0 \otimes 1$ generate $\tilde{M}$, we have $J \otimes_A \tilde{A} = J\tilde{A}$ is the annihilator of $\tilde{M}$. Now, the fibers over the faithfully flat extension $A/J \subseteq \tilde{A}/J\tilde{A}$ are just the fibers over $P \subseteq A$ for the inclusion $A \subseteq \tilde{A}$, for those primes P with $J \subseteq P$. Since the fibers of the inclusion $A \subseteq \tilde{A}$ are zero dimensional[3], it follows from what we have done in the previous section that $\dim(A?J) = \dim(\tilde{A}/J\tilde{A})$. Hence $\dim(M) = \dim(\tilde{M})$. Finally, the discussion in (i) above shows that

$$\lambda_{R(y)}(\tilde{M}_n) = \lambda_{R(y)}(R(y) \otimes_R M_n) = \lambda_R(M_n),$$

for all $n \geq 0$. This shows that in finding the Hilbert polynomial of a graded module, we may assume that the degree zero component of the underlying ring has an infinite residue field.

Here is the main theorem concerning Hilbert polynomials of $\mathbb{N}$-graded modules.

[3]If $p \subseteq A$ is a prime ideal, then the fiber over P in $\tilde{A}$ is just $k(p)(y)$.

Therem H5. *Let $A = \bigoplus_{n\geq 0} A_n$ be a finitely generated R-algebra, where $A_0 = (R, \mathfrak{m}, k)$ is a local Artinian ring with infinite residue field. We assume that A is a standard, graded R-algebra, i.e., $A = R[A_1]$. Let $M = \bigoplus_{n\geq 0} M_n$ be a finitely generated, graded A-module. Then $H_M(n) := \lambda_R(M_n) < \infty$, for all n and $H_M(n)$ agrees with a numerical polynomial $P_M(x)$ of degree $\dim(M) - 1$, for $n >> 0$.*

Proof. By the comments above, if need be, we may replace A by $R(y) \otimes_R A$ and M by $R(y) \otimes_R M$. This preserves the lengths and dimensions in question, so we may pass to $R(y)$ and upon changing notation assume that residue field of R is infinite. To see that $\lambda(M_n) < \infty$ for all n, note that $M_0 \oplus M_1 \oplus \cdots \oplus M_n = M/M_{\geq n+1}$ is a finite A-module annihilated by A_+^{n+1}. Thus, it is a finite A/A_+^{n+1}-module. This latter ring is finite over R, which implies that $M_0 \oplus M_1 \oplus \cdots \oplus M_n$ is a finite R-module. Thus, each M_j is a finite R-module, and therefore has finite length.

To show the existence of $P_M(n)$, we induct on $\dim(M)$. If $\dim(M) = 0$, then any prime ideal Q minimal over the annihilator of M is a maximal ideal. On the other hand, since M is a graded A-module, it associated primes are graded. Since $A_0 = R$ is local, $\mathcal{M}$ is the only graded maximal ideal.[4] Thus A_+ is contained in the annihilator of M, which implies $M_n = 0$, for $n >> 0$. Thus, we may take $P_M(x)$ to be the zero polynomial, which by standard convention has degree -1.

Now suppose $\dim(M) > 0$. Then $\text{Ass}(M) \neq \mathcal{M}$, so by Lemma G5, there exists $f \in A_1$ a superficial element on M. We will assume f has been chosen as in the proof of Lemma G5. Note that in this case, since $R = A_0$ is zero-dimensional, $\mathcal{M}$ is the only prime ideal in A containing A_+. Suppose $c > 0$ satisfies $(0 :_M f)_n = 0$, for all $n \geq c$. We have an exact sequence of graded A-modules

$$M \xrightarrow{\cdot f} M \to M/fM \to 0,$$

which induces an exact sequence of R-modules

$$M_{n-1} \xrightarrow{\cdot f} M_n \to (M/fM)_n \to 0,$$

for all n. Our choice of n implies that the sequence

$$0 \to M_{n-1} \xrightarrow{\cdot f} M_n \to (M/fM)_n \to 0,$$

is exact for all $n \geq c+1$. It follows that $H_M(n) - H_M(n-1) = H_{M/fM}(n)$, for all $n \geq c+1$. Our choice of f, and the fact that $\dim(M) > 0$, imply that f is not in any prime minimal over the annihilator of M, so that $\dim(M/fM) = \dim(M) - 1$.[5] By induction, $H_{M/fM}(n)$ agrees with a numerical polynomial $P_{M/fM}(x)$ of degree $\dim(M/fM) - 1$ for $n >> 0$. On the other hand, since $H_M(n) - H_M(n-1) = H_{M/fM}(n)$, for all $n \geq c+1$, by the second remark above concerning numerical polynomials, $H_M(n)$ agrees with a numerical polynomial, say $P_M(x)$, for $n >> 0$ whose degree equals $1+\text{degree}(P_{M/fM}(x))$. But $1+\text{degree}(P_{M/fM}(x)) = \dim(M) - 1$, which is what we want. ☐

Definition. The function $H_M(n)$ above is called the *Hilbert function* of M, while the polynomial $P_n(x)$ is called the *Hilbert polynomial* of M.

We now want to apply the theorem above to the associated graded ring of an $\mathfrak{m}$-primary ideal. Recall that the associated graded ring of a Noetherian ring R with respect to an ideal $I \subseteq R$,

$$\mathcal{G} := \bigoplus_{\mathfrak{n}\geq 0} I^n/I^{n+1} = \mathcal{R}/I\mathcal{R},$$

where $\mathcal{R}$ is the Rees ring of R with respect to I. Note that as an R/I-algebra, $\mathcal{G} = R/I[I/I^2]$, so that $\mathcal{G}$ is a standard graded, finitely generated R/I-algebra. If I is $\mathfrak{m}$-primary, then $R/I = \mathcal{G}_0$ is an Artinian ring, so Theorem H5 applies. We give two versions of the Hilbert polynomial associated to $\mathfrak{m}$-primary ideal I.

Corollary I5. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d and $I \subseteq R$ an $\mathfrak{m}$-primary ideal.*

(i) The function $\tilde{H}_I(n) := \lambda_R(I^n/I^{n+1})$ agrees with a numerical polynomial $\tilde{P}_I(x)$ of degree $d-1$, for $n >> 0$.

[4] If $J \subseteq A$ is a proper graded ideal, and $a \in J$, write $a = a_0 + a_1 + \cdots a_s$, with each $a_i \in J$. Then each $a_i \in J$. In particular $a_0 \in J$, and a_0 is not a unit so, $a_0 \in \mathfrak{m}$. Moreover, $a - a_0 \in A_+$, so $J \subseteq (\mathfrak{m}, A_+)A = \mathcal{M}$.

[5] Note that the primes containing the annihilator of M/fM are the primes containing f and the annihilator of M, which shows $\dim(M/fM) = \dim(M) - 1$.

(ii) The function $H_I(n) := \lambda(R/I^{n+1})$ agrees with a numerical polynomial $P_I(x)$ of degree d, for $n >> 0$.

Proof. The first statement is immediate from Theorem H5, since $\dim(\mathcal{G}) = d$. The second statement follows from the first, by our discussion of numerical polynomials, since $H_I(n) - H_I(n-1) = \tilde{P}_I(n)$, for $n >> 0$. $\square$

Definition. Maintaining the notation from Corollary I5, it follows from the discussion above on numerical polynomials, that we can write

$$P_I(x) = e_0\binom{x+d}{d} + e_1\binom{x+d-1}{d-1} + \cdots + e_{d-1},$$

where each $e_j \in \mathbb{Z}$ and $e_0 > 0$. The integer e_0 is called the *multiplicity of* I and is denoted $e(I)$. Note that if we write $P_I(x)$ in the form $q_0x^d + q_1x^{d-1} + \cdots + q_d$, with each $q_j \in \mathbb{Q}$, then $q_0 = \frac{e(I)}{(d)!}$. Thus,

$$e(I) = \lim_{n\to\infty} \frac{d!}{n^d} \cdot \lambda(R/I^{n+1}).$$

Note that $e(I)$ is also the normalized leading coefficient of $\tilde{P}_I(x)$, since $P_I(x) - P_I(x-1) = \tilde{P}_I(x)$.

For some of our results below, we will need refine the superficial element argument from above, which we do in the proposition below.

Definition. Let $I \subseteq R$ be a ideal in the Noetherian ring R. $a \in I$ is said to be a *superficial element for I* if there exists $c > 0$ such that $(I^n : a) \cap I^c = I^{n-1}$, for $n \geq c$.

Proposition J5. *Let $(R, \mathfrak{m}, k)$ be a local ring with infinite residue field and suppose $I \subseteq R$ is an ideal having height greater than zero. Then:*

(i) There exists $a \in I$, a superficial element for I, that is also a minimal generator for I.
(ii) If grade$(I) > 0$, then there exists a superficial element for I that is both a minimal generator for I and a non-zerodivisor.
(iii) If $a \in I$ is superficial for I, and a non-zerodivisor, then $(I^n : a) = I^{n-1}$, for $n >> 0$.

Proof. We will prove parts (i) and (ii) at the same time. Let $\mathcal{G}$ denote the associated graded ring of R. By proof of Lemma G5, if $a \in I/I^2$ has the property that its image $\overline{a}$ in $\mathcal{G}_1$ does not belong to any associated prime of (0) in $\mathcal{G}$ that contains $\mathcal{G}_+$, then, in the ring $\mathcal{G}$, $\overline{a}$ is superficial in the sense described there. Suppose $\overline{a}$ is such an element (which exists, by Lemma G5). Let c be as in Lemma G5. Then suppose $b \in (I^n : a) \cap I^c$, with $n \geq c$. If $b \notin I^{n-1}$, choose e maximal such that $e \geq c$, yet $e < n-1$ with $b \in I^e$. Then $\overline{b} \in \mathcal{G}_e$. Now, on the one hand, $\overline{a} \in \mathcal{G}_1$, so $\overline{a} \cdot \overline{b} \in \mathcal{G}_{e+1}$. But $ab \in I^n$ and $n > e+1$, so $\overline{a} \cdot \overline{b} = 0$ in $\mathcal{G}$. Since $e \geq c$, this means $\overline{b} = 0$, i.e., $b \in I^{e+1}$, contrary to the choice of e. Thus, in fact, $b \in I^{n-1}$, so a is a superficial element for I.

Now let us write $\mathcal{G} = \mathcal{R}/I\mathcal{R}$, where $\mathcal{R}$ is the Rees ring of R with respect to I. Then a primary decomposition of (0) in $\mathcal{G}$ corresponds to a primary decomposition of $I\mathcal{R}$. Let $Q_1, \ldots, Q_r$ be the associated primes in a primary decomposition of $I\mathcal{R}$ that do not contain $\mathcal{R}_+$. Let $J_i = \{a \in R \mid at \in Q_i\}$. Note that by definition, $I \nsubseteq J_i$, therefore $J_i \cap I$ is properly contained in I. Write $d := \dim_k(I/\mathfrak{m}I)$ and take $J = (a_2, \ldots, a_d)R$, where the images of the a_i in $I/\mathfrak{m}I$ are linearly independent. Then the subspaces $(J_1 + \mathfrak{m}I)/\mathfrak{m}I, \ldots, (J_r + \mathfrak{m}I)/\mathfrak{m}I, (J + \mathfrak{m}I)/\mathfrak{m}I$ are proper subspaces of the k-vector space $I/\mathfrak{m}I$. Take $a \in I$ such that its image in $I/\mathfrak{m}I$ does not belong to any of these subspaces.

Then, one the one hand, $a \notin J_i$ all i, so $at \notin Q_i$ all i. Thus, by our discussion above, a is a superficial element for I. On the other hand, since the image of a in $I/\mathfrak{m}I$ does not belong to $(J + \mathfrak{m}I)/\mathfrak{m}I$, the images of $a, a_2, \ldots, a_d$ in $I/\mathfrak{m}I$ are linearly independent over k, and thus form a minimal set of generators of I. In particular, a is a minimal generator of I. This gives (i). If in addition grade$(I) > 0$, let $P_1, \ldots, P_s$ denote the associated primes of R and set $W_i := P_i \cap I$, for each i. Then since $I \nsubseteq P_i$, W_i is properly contained in I. Thus the subspaces $(W_i + \mathfrak{m}I)/\mathfrak{m}I$ are proper subspaces of $I/\mathfrak{m}I$ and if a is chosen so that its image in $I/\mathfrak{m}I$ also avoids these subspaces, then $a \notin P_i$, for all i, and thus, we have that a is also a non-zerodivisor.

Finally, take $a \in I$ as in the statement of (iii). By the Artin-Rees lemma, there exists $k > 0$ such that $I^n \cap (a) \subseteq I^{n-k}$, for all $n \geq k$. Let c be as in (i). For any $n \geq c+k$, suppose $ra \in I^n$. Then $ra \in I^n \cap (a) \subseteq I^{n-k}a$. We can write $ra = ia$, with $i \in I^{n-k}$. Then $(r-i)a = 0$, and thus, $r = i$, since a is a non-zerodivisor. Therefore, $r \in I^{n-k} \subseteq I^c$, since $n \geq k+c$. Therefore $r \in (I^n : a) \cap I^c = I^{n-1}$, which is what we want. $\square$

Applications of Superficial Elements. We assume that $(R, \mathfrak{m}, k)$ is a local ring of dimension $d > 0$, and $I \subseteq R$ is an $\mathfrak{m}$-primary ideal.

(i) Suppose $a \in I$ is a superficial element and a non-zero divisor. Set $R^* = R/(a)$. Then $e(I) = e(IR^*)$. To see this, note that by (iii) in the proposition above, $(I^n : a) = I^{n-1}$, for n sufficiently large. Thus, the sequence

$$0 \to R/I^{n-1} \stackrel{\cdot a}{\to} R/I^n \to R/(I^n, a)R \to 0,$$

is exact for large n. Since $R/(I^n, a)R = R^*/I^nR^*$, we have

$$\lambda(R/I^n) - \lambda(R/^{n-1}) = \lambda(R^*/I^nR^*),$$

for $n >> 0$. Thus, $P_{IR^*}(x) = P_I(x) - P_I(x-1)$. Now, if $f(x)$ is a polynomial of degree d, then $f(x) - f(x-1)$ is a polynomial of degree $d-1$ whose leading coefficient is d times the leading coefficient of $f(x)$. Thus, the normalized leading coefficient of $P_{IR^*}(n)$ is $e(I)$, which gives $e(IR^*) = e(I)$.

(ii) This item shows that one can often assume that the ideal I has a superficial element that is a non-zerodivisor. Suppose $\text{depth}(R) = 0$, so that I does not contain a non-zerodivisor. Let L be the stable value of the increasing chain of ideals $(0 : I) \subseteq (0 : I^2) \subseteq \cdots$. Let's first note that $I^n \cap L = 0$, for $n >> 0$. Suppose $L = (0 : I^c)$. By the Artin-Rees lemma, there exists k such that $I^n \cap L \subseteq I^{n-k}L$. When $n - k \geq c$, $I^{n-1}L = 0$, which gives what we want. Now, for all n, we have an exact sequence

$$0 \to (I^n + L)/I^n \to R/I^n \to \tilde{R}/I^n\tilde{R} \to 0.$$

Thus $\lambda(R/I^n) = \lambda(\tilde{R}/I^n\tilde{R}) + \lambda((I^n + L)/I^n)$. However $(I^n + L)/I^n) \cong L/(I^n \cap L) = L$, for $n >> 0$. Thus, $P_I(x) = P_{I\tilde{R}}(x) + \lambda(L)$. Since $\dim(R) > 0$, $P_I(x)$ has degree greater than zero. Thus, the normalized leading coefficients of $P_I(x)$ and $P_{I\tilde{R}}(x)$ are the same, so that $e(I) = e(I\tilde{R})$. However, $I\tilde{R}$ has grade at least one. To see this, suppose $\text{grade}(I\tilde{R}) = 0$, Then there exists $0 \neq \tilde{r} \in \tilde{R}$ such that $\tilde{r} \cdot I\tilde{R} = 0$. Interpreting this in R, we have $rI \subseteq L$. Thus, $rI \cdot I^c = 0$. Therefore, $r \in (0 : I^{c+1}) = (0 : I^c) = L$, a contradiction. Therefore, $\text{grade}(I\tilde{R}) > 0$. This shows that we can always pass to a ring in which the multiplicity of I stays the same but the image of I has a superficial element that is a non-zerodivisor (when the residue field is infinite).

(iii) Assume I is generated by a system of parameters. Then $e(I) \leq \lambda(R/I)$. To see this, we may assume k is infinite. Now induct on d. Suppose $d = 1$, so $I = aR$. Let L be as in (ii). Then $e(aR) = e(a\tilde{R})$. Now in $\tilde{R}$, the image of a is a non-zerodivisor, so we we have that $\tilde{R}/(\tilde{a}) \cong a^{n-1}\tilde{R}/a^n\tilde{R}$, for all n. Applying this to the filtration $(0) \subseteq a^n\tilde{R} \subseteq a^{n-1}\tilde{R} \subseteq \cdots \subseteq \tilde{R}$, shows that $\lambda(\tilde{R}/a^n\tilde{R}) = \lambda(\tilde{R}/a\tilde{R}) \cdot n$, for all n. Thus, $\lambda(\tilde{R}/a\tilde{R}) = e(a\tilde{R}) = e(aR)$. Since $\lambda(\tilde{R}/a\tilde{R}) \leq \lambda(R/aR)$, we have $e(aR) \leq \lambda(R/aR)$, which is what we want. The inductive step is similar. Let L be as in (ii) and $\tilde{R} = R/L$. Then $e(I) = e(I\tilde{R})$ and $\lambda(\tilde{R}/I\tilde{R}) \leq \lambda(R/I)$. Thus, if we can prove the inequality we seek over $\tilde{R}$, it will hold in R. Note, that L is a nilpotent ideal, so that $\dim(\tilde{R}) = \dim(R)$, and hence $I\tilde{R}$ is generated by a system of parameters. Changing notation, we now assume that $\text{grade}(I) > 0$. Now, by Proposition J5, we may assume that the first generator, say a, of I is a superficial element and a non-zerodivisor. Setting $R^* := R/aR$, by (ii), we have $e(IR^*) = e(I)$. IR^* is generated by a system of parameters, so by induction $e(IR^*) \leq \lambda(R^*/IR^*)$. But $R/I \cong R^*/IR^*$, so $\lambda(R/I) = \lambda(R^*/IR^*)$, which completes the proof.

Remark. (i) For R and $I \subseteq R$ as above, suppose $a \in I$ is any element such that $\dim(R/aR) = \dim(R) - 1$, e.g., a is part of a system of parameters for R. Then the exact sequence

$$0 \to (I^{n+1} : a)/I^n \to R/I^n \stackrel{\cdot a}{\to} R/I^{n+1} \to R/(I^{n+1}, a) \to 0,$$

gives $\lambda(R/I^{n+1}) - \lambda(R/I^n) = \lambda(R/(I^{n+1}, a)) + \lambda((I^{n+1} : a)/I^n)$, which shows that $e(I/aR) \geq e(I)$. This clearly extends to R/J for any ideal J generated by part of a system of parameters, but does not extend to an arbitrary ideal J.

(ii) Since our main goal is the Rees multiplicity theorem, we are interested in the relevant results concerning the multiplicity of ideals in local rings. However, certain technical results needed along the way are made easier by extending the notion of multiplicity to modules. To that end, let $(R, \mathfrak{m})$ be a local ring of dimension d, $I \subseteq R$ an $\mathfrak{m}$-primary ideal, and M a finitely generated R-module. Then the module $\mathcal{M} := \bigoplus_{n \geq 0} I^nM/I^{n+1}M$ is a finitely generated $\mathcal{G}$-module. Thus, by Theorem H5, the lengths $\lambda_R(I^n/I^{n+1}M)$ agree with a numerical polynomial of degree $\dim(\mathcal{M}) - 1$, for $n >> 0$. In particular, this polynomial has degree less than or equal to $d-1$. It follows that the polynomial $P_{I,M}(x)$ that agrees with $\lambda(M/I^nM)$ for $n >> 0$ has degree less than or equal to d. This enables us to define a *multiplicity function* $e(I,-)$ as follows: $e(I,M) = \lim_{n\to\infty} \frac{d!}{n^d} \cdot \lambda(M/I^nM)$. Of course, $e(I,R) = e(I)$.

The next proposition plays a key role in the proof of the associativity formula for multiplicities. The associativity formula often allows one to reduce a question about the multiplicity of an ideal in a local ring to the same question when the ring is a domain. This will be especially important when we relate the multiplicity of an ideal to the Rees valuation rings associated to the ideal.

Proposition K5. *Let $(R, \mathfrak{m}, k)$ be a local ring, $I \subseteq R$ an $\mathfrak{m}$-primary ideal, and $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. Then $e(I,B) = e(I,A) + e(I,C)$.*

Proof. For all $n \geq 1$, we have an exact sequence

$$0 \to A/(I^nB \cap A) \to B/I^nB \to C/I^nC \to 0,$$

and thus

$$\lambda(B/I^nB) = \lambda(A/(I^nB \cap A)) + \lambda(C/I^nC), \qquad (*)$$

for all n. Now, by the Artin-Rees lemma there exists $c > 0$ such that $I^nB \cap A \subseteq I^{n-c}A$, for all $n > c$. Thus,

$$\lambda(A/I^{n-c}A) \leq \lambda(A/I^nB \cap A)) \leq \lambda(A/I^nA), \qquad (**)$$

for all $n > c$. Applying $\lim_{n\to\infty} \frac{d!}{n^d} \cdot \lambda(-)$ to (**) shows that $\lim_{n\to\infty} \frac{d!}{n^d} \cdot \lambda(A/(I^nB \cap A))$ exists and equals $e(I,A)$. Therefore, applying $\lim_{n\to\infty} \frac{d!}{n^d} \cdot \lambda(-)$ to equation (*) gives what we want. □

Corollary L5. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d, $I \subseteq R$ an $\mathfrak{m}$-primary ideal and M a finitely generated R-module. Then* degree $P_{I,M}(x) = \dim(M)$. *Thus, $e(I,M) = 0$ if and only if $\dim(M) < d$.*

Proof. The second statement follows immediately from the first. For the first statement, we may mod out the annihilator of M and assume that the annihilator of M is zero. This holds because neither the dimension of M nor the lengths of M/I^nM change when viewing M as an R-module or a module over R modulo its annihilator. We now have $\dim(M) = \dim(R)$ and an inclusion $R \hookrightarrow M \oplus \cdots \oplus M$, where this map is defined as in the second comment above concerning extending residue fields. Set $C := M \oplus \cdots \oplus M$, so that C is a finite R-module. It follows from the proposition above that $e(I,R) \leq e(I,C)$. Thus, the degree of $P_{I,C}(x)$ is greater than or equal to the degree of $P_I(x)$, which is $\dim(R)$. On the other hand, we always have that degree $P_{I,C}(x) \leq \dim(R)$, so equality holds. Since $P_{I,C}(x)$ and $P_{I,M}(x)$ have the same degree, the proof is complete. □

Remark. For the proof of the associativity formula, we need the following standard fact. If M is a finite module over the Noetherian ring R, then there exists a filtration $(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$ such that each quotient $M_i/M_{i-1} \cong R/P_i$, for some $P_i \in \mathrm{Spec}(R)$. To see this, let M' be maximal among all submodules of M admitting a filtration of the required type. The set of such modules in non-empty, since if $P \in \mathrm{Ass}(M)$, R/P is isomorphic to a submodule of M. If $M' \neq M$, then we can extend the filtration one step beyond M' by considering a submodule of M/M' corresponding to R/P, for $P \in \mathrm{Ass}(M/M')$.

Proposition M5. (Associativity formula) *Let $(R, \mathfrak{m}, k)$ be a local ring, $I \subseteq R$ an $\mathfrak{m}$-primary ideal and M a finitely generated R-module. Then*

$$e(I,M) = \sum_P e(I, R/P)\lambda(M_P),$$

where the sum is taken over all primes $P \in Spec(R)$ such that $dim(R/P) = (R)$. In particular,

$$e(I) = \sum_P e((IR+P)/P)\lambda(R_P).$$

Proof. Let $(0) = M_0 \subseteq M_1 \subseteq M_r = M$ be a filtration of M with each $M_i/M_{i-1} \cong R/Q_i$, for each $1 \leq i \leq r$, and $Q_i \in \mathrm{Spec}(R)$. Now, for each i, there is a short exact sequence

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0,$$

from which it follows that $e(I, M_i) = e(I, M_{i-1}) + e(I, M_i/M_{i-1})$. Putting these equations all together shows that $e(I,M) = \sum_i e(I, R/Q_i)$. By Corollary L5, $e(I, R/Q_i) \neq 0$ if and only if $\dim(R/Q_i) = \dim(R)$.

Thus, the only terms in the sum $e(I, M) = \sum_i e(I, R/Q_i)$ that are non-zero, are the terms for which $\dim(R/Q_i) = \dim(R)$. Suppose Q_i satisfies $\dim(R/Q_i) = \dim(R)$. Q_i may appear more than once. We note that the number of times R/Q_i appears in the filtration of M is $\lambda(M_{Q_i})$. Localize R and M at Q_i. Then M_{Q_i} is a finite length R_{Q_i}-module, and the original filtration localizes to a new filtration whose factors are just $k(Q_i)$, and none of the original factors corresponding to R/Q_i are lost. This new filtration is a composition series for M_{Q_i} and the number of factors is $\lambda(M_{Q_i})$, which gives what we want. Finally, if $P \in \operatorname{Spec}(R)$ satisfies $\dim(R/P) = \dim(R)$ and P does not appear in the filtration of M, then $M_P = 0$. So it can harmlessly be included in the sum $e(I, M) = \sum_i e(I, R/Q_i)$. This shows

$$e(I, M) = \sum_P e(I, R/P)\lambda(M_P),$$

where the sum is taken over all primes $P \in \operatorname{Spec}(R)$ such that $\dim(R/P) = \dim(R)$. □

The following proposition is another type of associativity formula, and is quite useful in calculating multiplicities. For this we need the following general observation.

Observation. Suppose $R \subseteq S$ are integral domains with quotient fields $K \subseteq L$, respectively. Assume S is finite over R, so that L is a finite extension of K. Then there exists a free R-module $F \subseteq S$ and a non-zero element $r \in R$ such that $rS \subseteq F$. To see this, let $U \subseteq R$ be the set of non-zero elements. Then S_U is finite, and hence integral, over $R_U = K$. Thus, S_U is a field. Since $S_U \subseteq L_U = L$, $S_U = L$. Thus, every element in the quotient field of S is a fraction of the form $\frac{s}{u}$, where $s \in S$ and $u \in R$. Now suppose $\frac{s_1}{u_1}, \dots, \frac{s_t}{u_t}$ form a basis for L over K. Then the R-module $F := Rs_1 + \cdots + Rs_t$ is a free R-module of rank t contained in S. On the other hand, if $s \in S$, we can write $s = \alpha_1 \frac{s_1}{u_1} + \cdots + \alpha_t \frac{s_t}{u_t}$, with each $\alpha_i \in K$. Clearly denominators shows that there exists $0 \neq r_0 \in R$ such that $r_0 s \in F$. Since S is a finite R-module, there exists $0 \neq r \in R$ with $rS \subseteq F$.

Proposition N5. *Let $(R, \mathfrak{m}, k)$ be a local domain with quotient field K and S an integral domain that is a finite R-module. Let L denote the quotient field of S, $\mathfrak{n}_1, \dots, \mathfrak{n}_r$ denote the maximal ideals of S, and set $k_i := S/\mathfrak{n}_i$. Then for any $\mathfrak{m}$-primary ideal $I \subseteq R$,*

$$e(I) \cdot [L : K] = \sum_{i=1}^{r} e(IS_{\mathfrak{n}_i})[k_i : k].$$

Proof. Let $F \subseteq S$ be as in the observation above and take $0 \neq r \in R$ such that $rS \subseteq F$. Then r annihilates S/F, and thus $\dim(S/F) < \dim(R)$. Therefore, $e(I, F/S) = 0$. Additivity of the multiplicity symbol applied to the exact sequence $0 \to F \to S \to S/F \to 0$, gives $e(I, S) = e(I, F)$. Here we are thinking of S and F as R-modules. Since F is free of rank $[L : K]$ over R, we have $e(IF) = e(I)[L : K]$. We must now show that $e(I, S) = \sum_{i=1}^{r} e(IS_{\mathfrak{n}_i})[k_i : k]$.

On the one hand, $e(I, S) = \lim_{n\to\infty} \frac{n^d}{d!}\lambda_R(S/I^nS)$. On the other hand, since $\lambda_R(k_i) = \lambda_k(k_i) = [k_i : k]$, if H is an $S_{\mathfrak{n}_i}$-module with finite length, additivity of the length function shows that $\lambda_R(H) = [k_i : k]\lambda_{S_{\mathfrak{n}_i}}(H)$. Thus, $\lambda_R(S_{\mathfrak{n}_i}/I^nS_{\mathfrak{n}_i}) = [k_i : k]\lambda_{S_{\mathfrak{n}_i}}(S_{\mathfrak{n}_i}/IS_{\mathfrak{n}_i})$. Since $I^nS = (I^nS_{\mathfrak{n}_1} \cap S) \cap \cdots \cap (I^nS_{\mathfrak{n}_r} \cap S)$, and the ideals $I^nS_{\mathfrak{n}_i} \cap S$ are co-maximal, we have an isomorphism of R-modules.

$$S/I^nS \cong S/(I^nS_{\mathfrak{n}_1} \cap S) \oplus \cdots \oplus S/(I^nS_{\mathfrak{n}_r} \cap S),$$

for all $n \geq 1$. Thus,

$$\lambda_R(S/I^nS) = \lambda_R(S/(I^nS_{\mathfrak{n}_1} \cap S)) + \cdots + \lambda_R(S/(I^nS_{\mathfrak{n}_r} \cap S)).$$

However, $\mathfrak{n}_i$ is the only maximal ideal of S containing $I^nS_{\mathfrak{n}_i} \cap S$, so that the ring $S/(I^nS_{\mathfrak{n}_i} \cap S)$ is local, i.e., $S/(I^nS_{\mathfrak{n}_i} \cap S) = S_{\mathfrak{n}_i}/I^nS_{n_i}$, for all i. Therefore,

$$\lambda_R(S/I^nS) = \lambda_R(S_{\mathfrak{n}_1}/I^nS_{\mathfrak{n}_1}) + \cdots + \lambda_R(S_{\mathfrak{n}_r}/I^nS_{\mathfrak{n}_r})).$$

From the first sentence of this paragraph we have

$$\lambda_R(S/I^nS) = [k_1 : k]\lambda_{S_{\mathfrak{n}_1}}(S_{\mathfrak{n}_1}/I^nS_{\mathfrak{n}_1}) + \cdots + [k_r : k]\lambda_{S_{\mathfrak{n}_r}}(S_{\mathfrak{n}_r}/I^nS_{\mathfrak{n}_r}).$$

multiplying this last equation by $\frac{n^d}{d!}$ and taking the limit as $n \to \infty$ gives $e(IS) = \sum_{i=1}^{r} e(IS_{\mathfrak{n}_i})[k_i : k]$, which is what we want. □

The following theorem due to DK provides a natural way to connect the multiplicity of an I-primary ideal to its Rees valuations rings. The result can be stated for rings that are not integral domains, but we will only need it in the domain case.

Theorem O5. *Let $(R, \mathfrak{m}, k)$ be a local domain of dimension at least two, and $I = (a_1, \ldots, a_d)R$ an ideal generated by a system of parameters. Set $T := R[\frac{a_1}{a_d}, \cdots, \frac{a_{d-1}}{a_d}]_{\mathfrak{m}R[\frac{a_1}{a_d}, \cdots, \frac{a_{d-1}}{a_d}]}$. Then $e(I) = e(IT)$.*

Proof. Without loss of generality, we may assume that k is infinite. We induct on $\dim(R)$. Suppose $\dim(R) = 2$. It follows from Proposition J5, that there exists $a_1' \in I$ such that a_1 is a minimal generator and a superficial element for I. Moreover, a_1' can be chosen to have the form $a_1 + ra_2$,for some $r \in R$. Note that $I = (a_1', a_2)$ and $R[\frac{a_1'}{a_2}] = R[\frac{a_1}{a_2}]$, so we may begin again assuming that a_1 is a superficial element for I.

We consider the natural homomorphism ϕ from the polynomial ring $R[x] \to R[\frac{a_1}{a_2}]$ taking x to $\frac{a_1}{a_2}$. As in the proof of Proposition D5, we let K be the kernel of this homomorphism, and $L := g(x)R[x]$, where $g = a_2x - a_1$. We still have that $a_2^c \cdot K \subseteq gR[x]$, for some c. Note that K_S is the kernel of ϕ_S which map obtained by inverting the elements S in $R[x]$, not in $\mathfrak{m}R[x]$. We will write $R(x)$ for $R[x]_S$ and note that ϕ_S maps $R(x)$ onto T.

We claim that g is superficial for $IR[x] = (g, a_2)R[x]$. To see this, suppose $c \geq 1$ satisfies $(I^n : a_1) = I^{n-1}$, for $n \geq c$. (Note that x_1 is a non-zerodivisor.) Suppose $f \cdot g \in I^nR[x]$, where $f = \sum_{i=0}^{s} r_ix^i$.

$$fg = -a_1r_0 + (a_2r_0 - a_1r_1)x + \cdots + (a_2r_{s-1} - a_1r_s)x^s + a_2r_2x^{s+1}.$$

It follows that $r_0 \in (I^n : a_1)$, so $r_0 \in I^{n-1}$. Therefore, $a_2r_0 \in I^n$, which implies, $r_1 \in (I^n : a_1)$. Thus, $r_1 \in I^{n-1}$. Inductively, we see that $r_i \in I^{n-1}$ for all i, so $f \in I^{n-1}R[x]$. Therefore, g superficial for $IR(x)$. Thus $e(I) = e(IR(x)) = e(IR(x)/gR(x))$. Set $A := R(x)/gR(x))$, so $e(I) = e(IA)$. Over A we have an exact sequence of A-modules $0 \to KA \to A \to T \to 0$. Note that a power of a_2 annihliates KA, and since a_2 is part of a system of parameters for A (in this case, an entire system of parameters), this means $\dim(KA) < \dim(A)$. Therefore, $e(IA, KA) = 0$. By the additivity of the multiplicity symbol, $e(IA, A) = e(I, T)$, which gives what we want.

Now suppose the result holds for local domains of dimension $d - 1$. As before, we may assume a_1 is superficial for I. Set $T_1 := R[\frac{a_1}{a_d}]_{\mathfrak{m}R[\frac{a_1}{a_d}]}$. Exactly the same proof as above shows that $e(I) = e(IT_1)$. The ring T_1 is a $(d-1)$-dimensional local ring with system of parameters $a_2, \ldots, a_d$. Thus, by induction $e(IT_1) = e(IT^*)$, where $T^* = T_1[\frac{a_2}{a_d}, \ldots, \frac{a_{d-1}}{a_d}]$. However, it is readily seen (as in the proof of Proposition E5) that

$$T^* = R[\frac{a_1}{a_d}, \cdots, \frac{a_{d-1}}{a_d}]_{\mathfrak{m}R[\frac{a_1}{a_d}, \cdots, \frac{a_{d-1}}{a_d}]} = T,$$

which completes the proof. □

The next two theorems are due to D. Rees. The proofs are due to DK. The proofs use the following standard fact, namely, that if I is an $\mathfrak{m}$-primary ideal in a local ring with infinite residue field, there exists an ideal $J \subseteq R$, generated by a system of parameters, such that $e(J) = e(I)$. This is essentially equivalent to the conclusion of Proposition F5, and is the form of Proposition F5 first given by Northcott and Rees. To see this, note that the proof of Proposition F5 shows that there exists an ideal $J \subseteq I$ generated by a system of parameters such that $JI^n = I^{n+1}$ for all n large, say $n \geq n_0$. Therefore, $J^{n-n_0}I^{n_0} = I^n$, for all $n > n_0$. It follows that for all $n > n_0$, $I^n \subseteq J^{n-n_0} \subseteq I^{n-n_0}$. Thus, for $n >> 0$, $P_I(n - n_0) \leq P_J(n) \leq P_I(n)$, which shows $e(J) = e(J)$.

In a similar vein, suppose J is an $\mathfrak{m}$-primary ideal and $a \in R$ is integral over J. Then there exists an equation $a^n + j_1a^{n-1} + \cdots + j_n = 0$, with each $j_s \in J^s$. This implies $a^n \in J(x, J)^{n-1}$, which gives $(a, J)^n = J(a, J)^{n-1}$. Thus, the same argument as above shows $e(a, J) = e(J)$. Since J is finitely generated, this shows $e(J) = e(\overline{J})$. Therefore, if $\overline{I} = \overline{J}$, for $\mathfrak{m}$-primary ideals $I, J \subseteq R$, then $e(I) = e(J)$.

Remark. Suppose V is a DVR with uniformizing parameter π and quotient field k. Then any non-zero element $a \in K$ can be written uniquely as $u\pi^n$, for some $n \in \mathbb{Z}$. This enables one to define a function $v : K \to \mathbb{Z} \cup \infty$ by $v(a) = n$, if $a \in K$ is non-zero and $a = u\pi^n$, and $v(0) = \infty$. The function v is called the *valuation associated to* V. If $J \subseteq V$ is an ideal, we write $v(J)$ for $v(a)$, where $J = aV$. This is the minimum value $v(j)$, with $j \in J$. Note that if $v(J) = e$, then $\lambda_V(V/J^n) = en$ for all n, so that $e(J) = e$, i.e., $e(J) = v(J)$, for all ideals $J \subseteq V$.

Theorem Q5. *Let $(R, \mathfrak{m}, k)$ be an analytically unramified local domain with infinite residue field and $I \subseteq R$ an $\mathfrak{m}$-primary ideal. Then there exist finitely many DVRs $V_1, \ldots, V_r$ between R and its quotient field, and finitely may positive integers $d_1, \ldots, d_r$ such that $e(I) = \sum_{i=1}^r d_i v_i(I)$, where v_i is the valuation associated to V_i.*

Proof. If we take $J \subseteq R$ such that J is generated by a system of parameters with $e(J) = e(I)$, we may replace J by I and begin again, assuming that $I = (a_1, \ldots, a_d)R$ is generated by a system of parameters. Taking T as in Theorem O5, we have $e(I) = e(IT)$. Now, by Rees's theorem on analytically unramified local domains, T' is a finite T-module. Since T and T' have the same quotient field, applying Proposition N5 gives $e(IT) = \sum_{i=1}^r d_i e(IV_i)$, where $d_i = [V_i/\mathfrak{m}_{V_i} : k]$. By the preceding remark, $e(IV_i) = v_i(I)$, which completes the proof. □

Remark. (i) Theorem Q5 was proven by Rees in a much more general form, essentially for any $\mathfrak{m}$-primary ideal in any local ring. However, one can achieve the general result by first extending the residue field, then passing to the completion of R, and modding out each minimal prime. The resulting rings are analytically unramified and quasi-unmixed. Rees' proof did not use the ring T, but rather used his earlier work on valuations associated to ideals (via the extended Rees ring).

(ii) The proof of Theorem Q5 shows that $e(I)$ is determined by the DVRs lying above the ring T, while Proposition E shows that R must satisfy the dimension formulas (equivalently, be quasi-unmixed) in order for the DVRs above T to determine the integral closure of I. This is one way to explain the need for the quasi-unmixed hypothesis in the multiplicity theorem of Rees.

We are now ready for the main theorem of this section.

Theorem R5. (Rees Multiplicity Theorem) *Let $(R, \mathfrak{m}, k)$ be a quasi-unmixed local ring, and $J \subseteq I$ $\mathfrak{m}$-primary ideal. If $e(J) = e(J)$, then $\overline{J} = \overline{I}$.*

Proof. We make a series of reductions. First, we may assume that k is infinite. This follows, since using $R(x)$ as before, $e(I) = e(IR(x))$ and $\overline{IR(X)} = \overline{I}R(x)$, and hence $\overline{IR(x)} \cap R = \overline{I}$, and similarly for J.

Since I and J are $\mathfrak{m}$-primary ideals, $\lambda(R/I^n) = \lambda(\widehat{R}/I^n\widehat{R})$ for all n, and similarly for J. Thus, $e(I\widehat{R}) = e(J\widehat{R})$. If $\overline{I\widehat{R}} = \overline{J\widehat{R}}$, then $\overline{I} = \overline{J}$, by the first part of the proof of Theorem M3. Thus we may may replace $\widehat{R}$ by R and assume that R is a complete, equidimensional local ring with infinite residue field.

Now, by the associativity formula,

$$e(I) = \sum_P e(I, R/P)\lambda(R_P) \quad \text{and} \quad e(J) = \sum_P e(J, R/P)\lambda(R_P),$$

where the sums are taken over all primes $P \in \operatorname{Spec}(R)$ such that $\dim(R/P) = \dim(R)$. Since R is equidimensional, this set of primes is excatly the set of minimal primes of R. Now, since $J \subseteq I$, $e(J, R/P) \geq e(I, R/P)$ for each term in the sums above. Since the two sums are equal, we must have $e(J, R/P) = e(I, R/P)$, for all minimal primes $P \subseteq R$. If the conclusion we seek holds over each R/P, then $\overline{J} = \overline{I}$, by Lemma B3. Thus, it is enough to prove the theorem under the further assumption that R is a complete local domain with infinite residue field.

Now, by Proposition F5 and the Remark preceding Theorem Q5, we may assume $J = (a_1, \ldots, a_d)$ is generated by a system of parameters. Set $T := R[\frac{a_1}{a_d}, \cdots, \frac{a_{d-1}}{a_d}]_{\mathfrak{m}R[\frac{a_1}{a_d}, \cdots, \frac{a_{d-1}}{a_d}]}$. By Theorem O5,

$$e(I) = e(J) = e(JT) \geq e(IT).$$

We claim that $e(IT) \geq e(I)$. To see this, consider the ring $S := R(x_1, \ldots, x_{d-1})$, the localization of $R[x_1, \ldots, x_{d-1}]$ at $\mathfrak{m}R[x_1, \ldots, x_{d-1}]$. Note that there is a natural ring homomorphism α from S onto T, which is just the localization of the homomorphism from Proposition D5. Like in the proof of Proposition D5, we set $L = (a_d x_1 - a_1, \ldots, a_d x_{d-1} - a_{d-1})S$, (though now S is the polynomial ring localized) and set K to be the kernel of α. Let $A := S/LS$. We now argue as in the proof of Theorem O5. Since L is generated by part of a system of parameters for S, $e(I) = e(IS) \leq e(IS/LS) = e(IA)$. On the other hand some power of a_d multiplies K into L, so that the image of this element in A annihilates KA. Thus $e(IA, KA) = 0$. Additivity of the multiplicity symbol shows that $e(IA, A) = e(IA, A/K) = e(IT)$. Thus, $e(I) \leq e(IT)$, which, now gives $e(JT) = e(IT)$.

Now we use the fact that R is a complete local domain, and therefore is analytically unramified and satisfies the dimension formula. Note that $JT = a_dT$. By the proof of Theorem Q5, if we write $Q_1, \ldots, Q_r$ for the maximal ideals of T' and set $V_i := T'_{Q_i}$, we have

$$e(JT) = \sum_{i=1}^{r} d_i v_i(J),$$

where the v_i are the valuations associated to V_i, and $d_i := [V_i/\mathfrak{m}_{V_i} : k]$. Moreover, by Proposition N5,

$$e(IT) = \sum_{i=1}^{r} d_i v_i(I).$$

Since J is contained in I, $v_i(I) \leq v_i(J)$, for all i, so by equality of the sums above, we have $v_i(I) = v_i(J)$, for all i. Thus, $JV_i = IV_i$, for all I. Therefore, $I \subseteq (JV_1 \cap R) \cap \cdots \cap J(V_r \cap R) = \overline{J}$, by Proposition E5, since R satisfies the dimension formula. Thus, $\overline{I} \subseteq \overline{J}$, and since the reverse containment always holds, $\overline{I} = \overline{J}$, and the proof is complete. □

6. Mixed multiplicities and Teissier's Theorem

The purpose of this section is to consider the following theorem of Teissier, which was given in a geometric setting.

Theorem A6. *Let $(R, \mathfrak{m}, k)$ be a reduced local ring of dimension $d \geq 2$ such k has characteristic zero, R is Cohen-Macaulay and also the localization of a finitely generated k-algebra at a maximal ideal. Let $I, J \subseteq R$ be two $\mathfrak{m}$-primary ideals. Then:*

$$e(IJ)^{\frac{1}{d}} \leq e(I)^{\frac{1}{d}} + e(J)^{\frac{1}{d}}.$$

This is similar in spirit to the Minkowski inequality from analysis which states that for $f, g \in L^p(\mathbb{R})$ (say), $||f+g||_p \leq ||f||_p + ||g||_p$, where $||f||_P = (\int |f|^p dx)^{\frac{1}{p}}$. A few years after Teissier's result was published, Rees and Sharp wrote a paper extending the result tow arbitrary local rings. We will present this result below. A key feature of Teissier's proof was the use of the so-called *mixed multiplicities* of I and J. These multiplicities are the normalized leading coefficients of the terms of total degree d in the Hilbert-Samuel polynomial that tracks the lengths of R/I^nJ^m, for $n, m >> 0$.

General Discussion. We begin with a general discussion of some of the things we will need for the Teissier-Rees-Sharp theorem.

(i) Just as in the one variable case, a numerical polynomial in two variables in a polynomial $P(x, y) \in \mathbb{Q}[x, y]$ such that $P(n, m) \in \mathbb{Z}$, for all $n, m \in \mathbb{Z}$ (or $\mathbb{N}$). Given $\mathfrak{m}$-primary ideals $I, J \subseteq R$, there exists a numerical polynomial of degree d $P_{I,J}(x, y) \in \mathbb{Q}[x, y]$ such that $P_{I,J}(n, m) = \lambda(R/I^nJ^m)$, for $n, m >> 0$. We will prove this below, but the proof of this fact is not much different from the proof of the one variable case, though below we will not discuss the general bi-graded case. Henceforth, we will think of this polynomial as a polynomial in n and m.

(ii) Because $P_{I,J}(n, m)$ is a numerical polynomial, there exist integer coefficients, e_{ij} such that

$$P_{I,J}(n, m) = \sum_{i+j \leq d} e_{i,j} \binom{n+i}{i} \binom{m+j}{j}.$$

The proof of this is almost exactly the same as in the one variable case because one can induct on the degree of the second variable, and mimic the previous proof. The integers e_{ij} such that $i + j = d$ are called the mixed multiplicities of I and J. We will see below that the mixed multiplicities are positive integers.

Moreover, using the binomial identities $\binom{n+i}{i} - \binom{n+i-1}{i} = \binom{n+i-1}{i-1}$, exactly the same proof from the previous section shows that if $H(n, m)$ is a numerical function, and $H(n, m) - H(n-1, m)$ agrees with a numerical polynomial of degree $d - 1$ for $n, m >> 0$, then $H(n, m)$ agrees with a numerical polynomial of degree d, for $n, m >> 0$.

(iii) It is easy to see that if we write out the terms of total degree d in $P_{I,J}(n,m)$, this expression can be written as

$$P_{I,J}(n,m) = \frac{1}{d!}\{e_0(I|J)n^d + \binom{d}{1}e_1(I|J)n^{d-1}m + \cdots + \binom{d}{d-1}e_{d-1}(I|J)nm^{d-1} + e_d(I|J)m^d\}.$$

Where $e_i(I|J) = e_{d-i,i}$, for $0 \leq i \leq d$.

(iv) Now suppose we fix $r, s \geq 1$. Then $e(I^rJ^s)$ is determined by the lengths $\lambda(R/(I^{rn}J^{sn})$ for $n >> 0$, which equal $P_{I,J}(rn, sn)$. If we substitute (rn, sn) into the equation in (iii) we get that the degree d term of $P_{I^rJ^s}(n)$ is

$$\frac{1}{d!}\{e_0(I|J)r^r + \binom{d}{1}e_1(I|J)r^{d-1}s + \cdots + \binom{d}{d-1}e_{d-1}(I|J)rs^{d-1} + e_d(I|J)s^d\}n^d.$$

This shows that

$$e(I^rJ^s) = e_0(I|J)r^d + \binom{d}{1}e_1(I|J)r^{d-1}s + \cdots + \binom{d}{d-1}e_{d-1}(I|J)rs^{d-1} + e_d(I|J)s^d.$$

Moreover, we also infer $e_i(I^r|J^s) = r^{d-i}s^ie_i(I|J)$, for all i. If we take $r = s = 1$ in the equation above, we have

$$e(IJ) = e_0(I|J) + \binom{d}{1}e_1(I|J) + \cdots + \binom{d}{d-1}e_{d-1}(I|J) + e_d(I|J). \qquad (\star)$$

(v) Let's see how the formula $(\star)$ might lead to the Minkowski-type inequality for multiplicities discovered by Teissier. Consider three positive integers a, b, c. If $a^{\frac{1}{d}} \leq b^{\frac{1}{d}} + c^{\frac{1}{d}}$, then raising this relation to the dth power, we get $a \leq \sum_{i=0}^{d}\binom{d}{i}b^{\frac{d-i}{d}}c^{\frac{i}{d}}$. It is clear that this last expression is equivalent to $a^{\frac{1}{d}} \leq b^{\frac{1}{d}} + c^{\frac{1}{d}}$. If we take $a = e(IJ)$, $b = e_0(I|J)$ and $c = e_d(I|J)$, then using $(\star)$, the Minkowski inequality for multiplicities holds if each $e_i(I|J) \leq e_0(I|J)^{\frac{d-i}{d}} \cdot e_d(I|J)^{\frac{i}{d}}$, for $1 \leq i \leq d-1$.

(vi) Set $e_i := e_i(I|J)$, for $0 \leq i \leq d$. Thus, Teissier's inequality holds if each $e_i^d \leq e_0^{d-i}e_d^i$. We claim these inequalities hold if $\frac{e_1}{e_0} \leq \frac{e_2}{e_1} \leq \cdots \leq \frac{e_d}{e_{d-1}}$. To see this, we first note that the required inequalities hold if and only if $(\frac{e_i}{e_0})^d \leq (\frac{e_d}{e_0})^i$, for all $1 \leq i \leq d-1$. We next note that

$$(\frac{e_i}{e_{i-1}})^{d-i}(\frac{e_{i-1}}{e_{i-2}})^{d-i}\cdots(\frac{e_1}{e_0})^{d-i} \leq (\frac{e_d}{e_{d-1}})^i \cdot (\frac{e_{d-1}}{e_{d-2}})^i \cdots (\frac{e_{i+1}}{e_i})^i,$$

since there are $(d-i)i$ factors on both sides of the inequality, and every factor on the left hand side is less than or equal to every factor on the right hand side. Multiply both sides of this last inequality by $(\frac{e_i}{e_{i-1}})^i(\frac{e_{i-1}}{e_{i-2}})^i\cdots(\frac{e_1}{e_0})^i$ to get

$$(\frac{e_i}{e_{i-1}})^d(\frac{e_{i-1}}{e_{i-2}})^d\cdots(\frac{e_1}{e_0})^d \leq (\frac{e_d}{e_{d-1}})^i \cdot (\frac{e_{d-1}}{e_{d-2}})^i \cdots (\frac{e_1}{e_0})^i.$$

Cancelling like numerators and denominators on both sides of the inequality gives $(\frac{e_i}{e_0})^d \leq (\frac{e_d}{e_0})^i$, which is what we want.

(vii) Thus, Tessier's theorem holds if one can show $\frac{e_1}{e_0} \leq \frac{e_2}{e_1} \leq \cdots \leq \frac{e_d}{e_{d-1}}$. However, these inequalities hold if $e_i^2 \leq e_{i-1}e_{i+1}$, for all $1 \leq i \leq d-1$.

(viii) Suppose $P(n,m)$ is a numerical polynomial in two variables of degree d and we write it in the form $P(n,m) = \sum_{i+j\leq d}\binom{n+i}{i}\binom{m+j}{j}$, then using the binomial identity $\binom{n+i}{i} - \binom{n-1+i}{i} = \binom{n-1+i}{i-1}$, we have that

$$P(n,m) = P(n-1,m) = \sum_{i+j\leq d-1} e_{i,j}\binom{n+i-1}{i-1}\binom{m+j}{j}.$$

Now suppose R^* is a local ring of dimension $d-1$ having the property that

$$P_{I,J}(n,m) - P_{I,J}(n-1,m) = P_{IR^*,JR^*}(n,m).$$

Ii follows that $e_i(I^*|J^*) = e_i(I|J)$, for $0 \leq i \leq d-1$. Similarly, if R' is a local ring of dimension $d-1$ having the property that

$$P_{I,J}(n,m) - P_{I,J}(n,m-1) = P_{IR',JR'}(n,m),$$

it follows that $e_i(I'|J') = e_i(I|J)$, for $1 \leq i \leq d$. Thus, if one can find rings R^* and R' satisfying these properties, once can prove the inequalities in (vii) by induction on the dimension.

(ix) An element $a \in I$ is said to be *superficial for* I, J if there exists $c \geq 1$ such that

$$(I^nJ^m : a) \cap I^cJ^m = (I^{n-1}J^m : a),$$

for all $n > c$ and $m \geq 1$. We will see below that superficial elements exists, and that for $R^* := R/aR$,

$$P_{I,J}(n,m) - P_{I,J}(n,m-1) = P_{I^*,JR^*}(n,m).$$

This reduces the proof of Minkowski inequality to the two-dimensional case.

(x) Let L denote the stable value of $(0 : \mathfrak{m}) \subseteq (0 : \mathfrak{m}^2) \subseteq \cdots$. Then, we have seen that the image of $\mathfrak{m}$ in $S := R/L$ has positive grade. Since I, J are $\mathfrak{m}$-primary, it follows that IS and JS also have positive grade. Now, exactly the same proof as in item (ii) of the Applications of Superficial Elements from the previous section shows that $P_{I,J}(n,m)$ and $P_{IS,JS}(n,m)$ differ by a constant. Since $\dim(R) = \dim(S) > 0$, this shows that $e_i(I|J) = e_i(IS, JS)$, for all i. Thus, we are free to assume I and J have positive grade when working with mixed multiplicities.

(xi) The discussion in the previous section concerning extending the residue field, in case k is finite, applies equally well in the current situations, so that the mixed multiplicities remain the same when we extend I and J to $R(x)$. Thus, one may harmlessly assume that k is infinite.

(xii) Let $K \subseteq I$ be a *reduction of* I, i.e., there exists n_0 such that $KI^{n_0} = I^{n_0+1}$. It follows that for all $n \geq n_0$, $K^{n-n_0}I^{n_0} = I^n$. Since $I^nJ^m \subseteq K^{n-n_0}J^m \subseteq I^{n-n_0}J^m$, for $n \geq n_0$, it follows that

$$P_{I,J}(n,m) \leq P_{K,J}(n-n_0,m) \leq P_{I,J}(n-n_0,m),$$

for $n \geq n_0$, and from this it follows that $e_i(I|J) = e_i(K|J)$, for all i (since the mixed multiplicities are positive). The proof of Theorem F5 shows that when k is infinite, there exists an ideal $K \subseteq I$, generated by a system of parameters such that K is a reduction of I. Thus, when k is infinite, we may replace I by a system of parameters and not change the mixed multiplicities.

(xiii) $e_0(I|J) = e(I)$ and $e_d(I|J) = e(J)$. To see this, take n, m sufficiently large, so that the lengths $\lambda(R/I^nJ^m) = P_{I,J}(n,m)$. Now, fix $m = m_0$. Then

$$\lambda(R/I^nJ^{m_0}) = \frac{1}{d!}\{e_0(I|J)n^d + \binom{d}{1}e_1(I|J)n^{d-1}m_0 + \cdots + \binom{d}{d-1}e_{d-1}(I|J)nm_0^{d-1} + e_d(I|J)m_0^d\} + \cdots$$

This shows that the lengths $\lambda(R/I^nJ^{m_0})$ are given by a polynomial of degree d whose normalized leading coefficient is $e_0(I|J)$. On the other hand, $\lambda(R/I^nJ^{m_0}) = \lambda(R/J^{m_0}) + \lambda(J^{m_0}/I^nJ^{m_0})$, for all n. Therefore, the degree d polynomial giving the lengths of $J^{m_0}/I^nJ^{m_0}$, for n large, also has $e_0(I|J)$ as its normalized leading coefficient. In other words, $e_0(I|J) = e(I, J^{m_0})$, when we regard J^{m_0} as an R-module. Since the annihilator of J^{m_0} has to be nilpotent, $\dim(J^{m_0}) = d$, as an R-module. Since $\dim(R/J^{m_0}) = 0$, additivity of the multiplicity symbol $e(I, -)$ applied to the exact sequence

$$0 \to J^{m_0} \to R \to R/J^{m_0} \to 0,$$

shows that $e(I, J^{m_0}) = e(I, R) = e(I)$. Thus, $e_0(I|J) = e(I)$, as required. The proof that $e_d(I|J) = e(J)$ is similar.

Our first goal is to show the existence of $P_{I,J}(n,m)$ while at the same time showing that for all $0 \leq i \leq d$, $e_i(I|J) > 0$. For this, we need superficial elements relative to a pair of ideals. The proof of this case is very similar to the proof of the superficial element proposition from the previous section, though there is an interesting wrinkle in the proof that does not occur with one ideal. We also need a bigraded version of the Artin-Rees Lemma. We just indicate the proof in the case we need because it is essentially the same as in the usual case.

Proposition B6. *Suppose $I, J, K \subseteq R$ are ideals. Then there exists $u, v \geq 1$ such that for all $n \geq u$ and $m \geq v$, $I^nJ^m \cap K \subseteq I^{n-u}J^{m-v}K$.*

Sketch of Proof. One uses the bigraded Rees algebra $\mathcal{R} := R[It, Js] = \bigoplus_{n,m\geq 0} I^nJ^mt^ns^m$ and considers the ideal $\mathcal{K} = \bigoplus(K \cap I^nJ^m)t^ns^m \subseteq \mathcal{R}$. This is a homogenous ideal with respect to the bigrading on $\mathcal{R}$, so it has a set of homogeneous generators. Take u greater than any exponent of t among these generators and v greater than the exponent of s among these generators. Then for any element $b \in I^nJ^m \cap K$ with $n \geq u$ and $m \geq v$, $bt^ns^m \in \mathcal{K}$. Write this element in terms of the generators of $\mathcal{K}$, and read off the homogeneous coefficients of the generators, to get the desired conclusion. □

Proposition C6. *Let $(R, \mathfrak{m}, k)$ be a local ring with infinite residue field and positive depth. Let $I, J \subseteq R$ be $\mathfrak{m}$-primary ideals. Then there exist $c > 0$ and a non-zerodivisor $a \in I$, that is also a minimal generator of I, such that $(I^nJ^m : a) = I^{n-1}J^m$, for all $n > c$ and all $m \geq 1$.*

Proof. We use the Rees ring $\mathcal{R}$ of R with respect to I and J. This is the ring $\mathcal{R} := R[It, Js]$, where t, s are indeterminates over R. This is a bi-graded R-algebra, generated in degrees (1,0) and (0,1) over R. Let $Q_1, \ldots, Q_r$ be the associated primes of $I\mathcal{R}$ not containing It and $Q_{r+1}, \ldots, Q_h$ be the remaining associated primes of $I\mathcal{R}$. Choose $c_0 > 0$ such that $(It)^{c_0}$ is contained in the Q_i-primary component of $I\mathcal{R}$, for $r + 1 \leq i \leq h$. We set $J_i := \{r \in R \mid rt \in Q_i\}$, for $r + 1 \leq i \leq h$. Then as before, $I \not\subseteq J_i$, so that $I \cap J_i$ is properly contained in I. Let $P_1, \ldots, P_b$ be the associated primes of R, and note that $W_i := I \cap P_i$ is properly contained in I. Finally, take $J \subseteq I$ so that $(J + \mathfrak{m}I)/\mathfrak{m}I \subseteq I/\mathfrak{m}I$ has dimension one less that the dimension of the vector space $I/\mathfrak{m}I$. Then the subspaces $(J_i + \mathfrak{m}I)/\mathfrak{m}I, (W_i + \mathfrak{m}I)/\mathfrak{m}I, (J + \mathfrak{m}I)/\mathfrak{m}I$ are all proper subspaces of $I/\mathfrak{m}I$, so there exists $a \in I$ whose image in $I/\mathfrak{m}I$ avoids these subspaces. Thus a is a minimal geberator of I and also a non-zerodivisor.

Now suppose $n > c_0$ and $r \in (I^nJ^m : a) \cap I^{c_0}J^m$. Then $rt^{c_0}s^m \in \mathcal{R}$. If $n = c_0 + 1$, then $c_0 = n - 1$, so $r \in I^{n-1}J^m$, which is what we want. Suppose $n > c_0 + 1$. Then $rt^{c_0}s^m \cdot at \in I^nJ^mt^{c_0+1}s^m \in I\mathcal{R}$. By definition of c_0, $rt^{c_0}J^m$ belongs to every Q_i-primary component of $I\mathcal{R}$, with $r + 1 \leq i \leq h$. On the other hand, the choice of a forces $rt^{c_0}s^m$ to be in the Q_j-primary components of $I\mathcal{R}$, for $1 \leq i \leq r$. Thus, $rt^{c_0}s^m \in I\mathcal{R}$. This implies $r \in I^{c_0+1}J^m$. We may repeat the argument until we arrive at $r \in I^{n-1}J^m$.

Finally, let u, v be chosen so that $(a) \cap I^nJ^m \subseteq aI^{n-u}J^{m-v}$, for $n \geq u$ and $m \geq v$. Take p such that $I^p \subseteq J$, so that $I^{pv} \subseteq J^v$. Now suppose $n > pv + u + c_0$ and take $r \in (I^nJ^m : a)$. Then $ra \in I^nJ^m \subseteq I^{u+c_0}J^{m+v}$ (the aforementioned wrinkle), so $ra \in (a) \cap I^{u+c_0}J^{m+v}$ and we can write $ra = ax$, for $x \in I^{c_0}J^m$. Thus, $r = x \in I^{c_0}J^m$, since a is a non-zerodivisor. Therefore, $r \in (I^nJ^m : a) \cap I^{c_0}J^m = I^{n-1}J^m$, by the previous paragraph. Taking $c := pv + u + c_0$ shows that $(I^nJ^m : a) = I^{n-1}J^m$, for all $n > c$, and all $m \geq 1$. □

We now show the existence of the Hilbert-Samuel polynomial associated to two $\mathfrak{m}$-primary ideals.

Theorem D6. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d, and $I, J \subseteq R$ $\mathfrak{m}$-primary ideals. Then there exists a numerical polynomial $P_{I,J}(n, m)$ of degree d such that $\lambda(R/I^nJ^m) = P_{I,J}(n, m)$, for $n, m >> 0$. Moreover, if we write the terms of total degree d in $P_{I,J}(n, m)$ as*

$$\frac{1}{d!}\{e_0(I|J)n^d + \binom{d}{1}e_1(I|J)n^{d-1}m + \cdots + \binom{d}{d-1}e_{d-1}(I|J)nm^{d-1} + e_d(I|J)m^d\},$$

then each $e_i(I|J) > 0$.

Proof. We use $H_{I,J}(n, m)$ to denote $\lambda(R/I^nJ^m)$ for all n, m. Without loss of generality, we may assume the residue field of R is infinite. We now induct on d. If $\dim(R) = 0$, then the conclusion of the theorem is clear. Assume $d > 0$. By (x) in the General Discussion above, and its predecessor in the previous section, if we write $S := L$, for $L := (0 : \mathfrak{m}^t)$, for $t >> 0$, the lengths $H_{I,J}(n, m)$ and the lengths $\lambda(S/I^nJ^mS)$ differ by a constant for large n, m. It follows that we may assume that R has positive depth, and hence I and J have positive grade. By the previous proposition, there exists $c > 0$ and $a \in I$, a non-zero divisor, such that $(I^nJ^m : a) = I^{n-1}J^m$, for all $n > c$ and $m \geq 1$. Set $R^* := R/aR$. For $n > c$, the exact sequence

$$0 \to R/I^{n-1}J^m \xrightarrow{\cdot a} R/I^nJ^m \to R^*/I^nJ^mR^* \to 0,$$

gives

$$H_{I,J}(n, m) - H_{I,J}(n - 1, m) = \lambda(R^*/I^nJ^mR^*).$$

Since $\dim(R^*) = d - 1$, by induction on d, the lengths of $R^*/I^nJ^mR^*$ agree with a polynomial of degree $d-1$, all of whose top coefficients are positive. By comment (ii) in the General Discussion above, $H_{I,J}(n, m)$ agrees with a polynomial numerical polynomial $P_{I,J}(n, m)$ of degree d, for $n, m >> 0$. If we write the terms of degree d in $P_{I,J}(n, m)$ as in the statement of the theorem, item (viii) in the discussion above shows

$e_i(I|J) = e_i(IR^*|JR^*) > 0$, for $0 \leq i \leq d-1$. Since $e_d(I|J) = e(J) > 0$ (by part (xiii)), the proof is complete. $\square$

We need two preliminary results before proving the Teissier-Rees-Sharp theorem. The first is a very special case of a general result known as *Lech's Lemma* and the second result is the key step in the Rees-Sharp proof.

Proposition D6. Let $(R, \mathfrak{m}, k)$ be a two-dimensional local ring with infinite residue and $I = (a, b)R$ an ideal generated by a system of parameters. Assume that $a \in I$ is a non-zerodivisor and a superficial element for I. Then,

$$e(I) = \lim_{n\to\infty} \frac{1}{n^2} \cdot \lambda(R/(a^n, b^n)R).$$

Proof. Let's first note that $e(I^n) = e((a^n, b^n)R)$, for all n. To see this, let $a^jb^j \in I^n$ be a monomial generator of degree n. Then $(a^ib^j)^n = (a^n)^i(b^n)^j \in (a^n, b^n)^nR$. This shows a^ib^j is integral over (a^n, b^n), and thus I^n and $(a^n, b^n)R$ have the same integral closure, and thus, the same multiplicity. Therefore,

$$n^2e(I) = e(I^n) = e((a^n, b^n)R) \leq \lambda(R/(a^n, b^n)R) \leq n \cdot \lambda(R/(a, b^n)R).$$

Here we are using the fact that if $I \subseteq R$ is an $\mathfrak{m}$-primary ideal in a local ring of dimension d, then for any $r \geq 1$, $e(I^r) = r^de(I)$.[6] Dividing the displayed equation by n^2 and taking the limit as $n \to 0$ we have,

$$e(I) \leq \lim_{n\to\infty} \frac{1}{n^2} \cdot \lambda(R/(a^n, b^n)R) \leq \lim_{n\to\infty} \frac{1}{n} \cdot \lambda(R/(a, b^n)R) = e(b, R/aR) = e(I/aR) = e(I),$$

since $a \in I$ is a superficial, non-zerodivisor.

Theorem E6. *Let $(R, \mathfrak{m}, k)$ be a two-dimensional local ring and $I, J \subseteq R$ $\mathfrak{m}$-primary ideals. Then,*

$$e(IJ) \leq 2e(I) + 2e(J).$$

Proof. We may assume that the residue field of R is infinite. We may also by modding out the stable value of $(0 : \mathfrak{m}^t)$, we may assume I, J have positive grade. By Theorem F5, there exists an ideal $K \subseteq I$ generated by a system of parameters with $\overline{K} = \overline{I}$. By the comments following Theorem O5, $e(K) = e(I)$. On the other hand, one also has $\overline{KJ} = \overline{IJ}$, and thus, $e(KJ) = e(IJ)$. Therefore, we may replace I by K, then change notation to assume that $I = (a, b)R$ is generated by a system of parameters. From our work in the previous section, we may further assume that $I = (a, b)$ with a a non-zerodivisor and a superficial element for I.

Let $F = R^2$, and observe that for all $n \geq 1$, we have a surjective R-module map $F/J^nF \to (a^n, b^n)R/(a^n, b^n)J^n$. Thus,

$$2 \cdot \lambda(R/J^n) \geq \lambda\{(a^n, b^n)R/(a^n, b^n)J^n\}.$$

Therefore,

$$\lambda(R/(a^n, b^n)) + 2\lambda(R/J^n) \geq \lambda(R/(a^n, b^n)) + \lambda\{(a^n, b^n)R/(a^n, b^n)J^n\} = \lambda(R/(a^n, b^n)J^n) \geq \lambda R/(I^nJ^n).$$

If we multiply the left hand side of this inequality by $\frac{2}{n^2}$ and take the limit as $n \to \infty$, we get $2e(I) + 2e(J)$ (using Lech's lemma on the first term). Multiplying the far right side of the inequality by $\frac{2}{n^2}$ and taking the limit as $n \to \infty$ gives, $e(IJ)$, which completes the proof. $\square$

We now have all of the pieces required to prove the main result of this section.

Proof of Theorem A6. We may assume the residue field of k is infinite, and proceed by induction on $d := \dim(R)$. We let $e_i(I|J)$ denote the mixed multiplicities of I and J, and set $e_i := e_i(I|J)$. By item (iv) of the General Discussion, we need to prove that $e_i^2 \leq e_{i-1}e_{i+1}$, for all $1 \leq i \leq d-1$. Suppose $d = 2$. By item (vii) of the General Discussion above, we must prove $e_1^2 \leq e_0e_2$. By item (iv) in the general discussion, for all $r, s \geq 1$, we have

$$e(I^rJ^s) = e_0r^2 + 2e_1rs + e_2s^2.$$

On the other hand, by Theorem E6,

$$e(I^rJ^s) \leq 2e(I^r) + 2e(J^s) = 2r^2e(I) + 2s^2e(J).$$

Thus,

$$e_0r^2 + 2e_1rs + e_2s^2 \leq 2e_0r^2 + 2e_2s^2,$$

[6] To see this, note that $\lambda(R/(I^r)^n) = P_I(rn)$, for $n >> 0$, which shows that the normalized leading coefficient of $P_{I^r}(n)$ is $r^de(I)$.

for all $r, s \geq 1$. Therefore,

$$0 \leq e_0 r^2 - 2e_1 rs + e_2 s^2,$$

for all r, s. If we substitute $r = e_1$ and $s = e_0$ into this last expression, we get

$$0 \leq e_0 e_1^2 - 2e_1^2 e_0 + e_2 e_0^2 = -e_0 e_1^2 + e_0^2 e_2.$$

Since $e_0 > 0$, , we can divide by e_0 and conclude $e_1^2 \leq e_0 e_2$.

Now suppose $d \geq 3$. By item (x) in the General Discussion, we may assume that I, J have positive grade. By Proposition C6, there exists $a \in I$, a non-zerodivisor that is superficial for the pair I, J. Set $R^* := R/aR$, so $\dim(R^*) = d - 1$. Then for $n >> 0$, we have an exact sequence

$$R/I^{n-1}J^m \xrightarrow{\cdot a} R/I^n J^m \to R^*/I^n J^m R^* \to 0.$$

It follows that for $n, m >> 0$, $P_{I,J}(n, m) - P_{I,J}(n-1, m) = P_{IR^*,JR^*}(n, m)$. By item (viii) in the General Discussion, $e_i(I|J) = e_i(IR^*, JR^*)$, for $0 \leq i \leq d-1$. Therefore, by induction, we have $e_i^2 \leq e_{i-1}e_{i+1}$, for all $1 \leq i \leq d-2$. Since the argument is symmetric in I and J, we may take $b \in J$ superficial for I, J and repeat what we have just done with the roles of I and J reversed to pick up the last relation $e_{d-1}^2 \leq e_{d-2}e_d$. □

Final Remarks. What about equality in the Minkowski inequality for multiplicities? It turns out that this is closely related to the integral closure of powers of ideals. Two ideals $I, J \subseteq R$ are said to be *projectively equivalent* if there exist positive integers $a, b \geq 1$ such that $\overline{I^a} = \overline{J^b}$. We first note that if this condition holds, then equality in the Minkowski inequality for multiplicities is more or less a formal consequence of the rules for manipulating the mixed multiplicities. To see this, we need to observe that if $L, K \subseteq R$ have the same integral closure, then $e_i(L|K) = e(L) = e(K)$, for all i. To see this, for one, we know from the previous section that $e(L) = e(K)$. We may also assume $L = K$, by item (xii) in the General Discussion. So suppose $L = K$. Then when we calculate $P_{L,K}(n, m) = P_{L,L}(n, m)$ we are calculating the lengths of $R/L^n K^m$, with n, m independent. But this is the same as considering the lengths of R/L^{n+m} as a function of two variables. Thus, if expand $P_L(n + m)$ out as function of n, m and compare the leading coefficients with those of $P_{L,L}(n, m)$, we see that $e_i(L, L) = e(L)$, for all i.

Now, if we trace through the sequence of steps that led from the Minkowski inequality to the set of inequalities $e_i^2 \leq e_{i-1}e_{i+1}$, we see two things: (i) The Minkowski inequality holds if and only if the set of inequalities $e_i^2 \leq e_{i-1}e_{i+1}$ hold and (ii) Equality in the Minkowski inequality holds if and only if $e_i^2 = e_{i-1}e_{i+1}$, for all i. Now, suppose $I, J \subseteq R$ are projectively equivalent, i.e., there exist $a, b \geq 1$ such that $\overline{I^a} = \overline{J^b}$. Then, all of the mixed multiplicities $e_i(\overline{I^a}|\overline{J^b})$ are equal, and consequently, all of the mixed multiplicities $e(I^a|J^b)$ are equal. Thus $e_i(I^a|J^b)^2 = e_{i-1}(I^a|J^b)e_{i+1}(I^a|J^b)$, for all i. However, from item (iv) in the General Discussion we have,

$$e_i(I^a|J^b)^2 = (a^{d-i}b^i)^2 e_i(I|J)^2 \text{ and } e_{i-1}(I^a|J^b)e_{i+1}(I^a|J^b) = (d^{d-i+1}b^{i-1}e_{i-1}(I|J)) \cdot (a^{d-(i+1)}b^{i+1}e_{i+1}(I|J)),$$

from which it follows that $e_i(I|J)^2 = e_{i-1}(I|J)e_{i+1}(I|J)$, for all i, and thus, equality holds in the Minkowski inequality.

In the geometric setting, Teissier proved the converse, which turns out to be a generalization of the Rees multiplicity theorem. In other works, the converse states that equality in the Minkowski inequality implies that the ideals are projectively equivalent. Using geometric techniques, Teissier reduced the question to surfaces, and used resolutions of singularities to finish off the proof. Rees and Sharp proved an algebraic version of this for two-dimensional quasi-unmixed local rings and DK showed how to reduced the general algebraic case to the two-dimensional case. A consequence of this theorem is that one gets a version of the Rees multiplicity theorem, without assuming a containment relation between I and J. The statement in this case would be: Let $(R, \mathfrak{m}, k)$ be a quasi-unmixed local ring of dimension d and $I, J \subseteq R$, $\mathfrak{m}$-primary ideals. If $e_0(I|J) = e_1(I|J) = \cdots = e_d(I|J)$, then $\overline{I} = \overline{J}$. The point is that if all of the mixed multiplicities are equal, it is not hard to see that equality must hold in the Minkowski inequality. Thus, $\overline{I^a} = \overline{J^b}$, for some a, b. But then $a^d e(I) = b^d e(J)$, and since $e(I) = e(J)$, $a = b$, so $\overline{I} = \overline{J}$.

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